Some results on α-quantiles

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Abstract

The aim of the paper is to present a simple method to rederive some recent results concerning distribution identities for the so called α-quantile for a real-valued stochastic process (see [1] and [2]). Also, the first discrete time version of this result, which seems due to [3], shall appear in continuous time. We first propose a (new) theorem presenting an upper and lower bound for the α-quantile for an arbitrary piecewise continuous real-valued function. The bounds are identical if the function is linear, so if the function represents sample paths for a stochastic process having stationary and independent increments, we can use the similar linearity in distribution to rederive the fundamental distribution equality as studied in [1].

Keywords: Gaussian process, renewal reward process, distribution equality and inequality.

1 Introduction

For some real-valued function $Y = (Y(t))_{t\geq 0}$, and a fixed $0 < T < \infty$, the quantity

$$Q_Y(\alpha, T) = \inf\left\{x \in \mathcal{R} :\ \frac{1}{T}\int_0^T \mathbf{1}(Y(s) \leq x)ds > \alpha\right\}, \quad 0 \leq \alpha < 1,$$

where $\mathbf{1}(C)$ is the indicator for a set C, and $\mathcal{R}$ the real line, is called the α-quantile for Y. If Y is some stochastic process with sample paths that are right-continuous with left-hand limits, there has recently been some interest in finding the distribution of $Q_Y(\alpha, T)$, see references, where the basic result seems to be in [3]. One interest today for studying α-quantiles is from option pricing, a major issue in financial mathematics.

The tool in this paper is to characterize $Q_Y(\alpha, T)$ more closely in terms of Y, see Theorem 2.1, and use this to correct and rederive in a simple and unified manner known results obtained in [1]-[3], see (2.3)-(2.9). The proof of Theorem 2.1 is simple, and derives an upper and lower bound for $Q_Y(\alpha, T)$. Finally, we give an example to indicate the difficulties of obtaining distribution equalities when 'nice' properties of Y are abandoned, and we obtain only some distribution inequalities.

Define

$$M_Y(T, x) = \frac{1}{T}\int_0^T \mathbf{1}(Y(s) \leq x)ds\,, \quad x \in \mathcal{R}\,, \tag{1.1}$$

and in the sequel we write Q_Y instead of $Q_Y(\alpha, T)$ whenever the value of α is not specific.

2 Characterization of α-quantiles

For any real-valued function $Y = (Y(t))_{t\geq 0}$ that is right-continuous with left-hand limits, it is readily checked that

$$Q_Y(0, T) = \inf_{[0,T]} Y(s)\,,$$

so the more challenging part is to understand $Q_Y(\alpha,T)$ for $\alpha>0$. If Y is continuous we can also directly see that

$$Q_Y \leq \sup_{[0,\alpha T]} Y(s)\,,$$

since $M_Y(T,x)>\alpha$, for any $x>\sup_{[0,\alpha T]} Y(s)$.

The starting point is to prove the following theorem.

Theorem 2.1 *Let $Y=(Y(t))_{t\geq 0}$ be some real-valued function which is right-continuous with left-hand limits. Then for $\alpha\in[0,1)$, Q_Y satisfies*

$$\sup_{\varepsilon\in(0,\alpha T]}\left\{\inf_{s\in[0,(1-\alpha)T)} Y(s+\varepsilon)\right\} \leq Q_Y \leq \inf_{\varepsilon\in[0,(1-\alpha)T)}\left\{\sup_{s\in(0,\alpha T]} Y(s+\varepsilon)\right\}. \tag{2.1}$$

Proof: Let first ε be such that $0\leq\varepsilon<(1-\alpha)T$, and split $M_Y(T,x)$ as

$$M_Y(T,x)=\frac{1}{T}\int_{\varepsilon}^{\varepsilon+\alpha T}\mathbf{1}(Y(s)\leq x)ds+\frac{1}{T}\int_{(0,T]\setminus(\varepsilon,\varepsilon+\alpha T]}\mathbf{1}(Y(s)\leq x)ds\,.$$

For $x>\sup_{s\in(\varepsilon,\varepsilon+\alpha T]}Y(s)$, we see that the first integral is equal to α, and since also $Y(\varepsilon+\alpha T)<x$, the right-continuity together with $\varepsilon<(1-\alpha)T$, implies that $Y(s)<x$, for s in some interval to the right of $\varepsilon+\alpha T$. Hence the second integral is positive, and therefore $M_Y(T,x)>\alpha$, so by the definition of Q_Y and since x was arbitrary, we have

$$Q_Y\leq\sup_{s\in(\varepsilon,\varepsilon+\alpha T]}Y(s)\,, \tag{2.2}$$

and consequently since ε is arbitrary

$$Q_Y\leq\inf_{\varepsilon\in[0,(1-\alpha)T)}\left\{\sup_{s\in(\varepsilon,\varepsilon+\alpha T]}Y(s)\right\}.$$

Let now ε satisfy $0<\varepsilon\leq\alpha T$, and split instead as

$$M_Y(T,x)=\frac{1}{T}\int_{\varepsilon}^{\varepsilon+(1-\alpha)T}\mathbf{1}(Y(s)\leq x)ds+\frac{1}{T}\int_{(0,T]\setminus(\varepsilon,\varepsilon+(1-\alpha)T]}\mathbf{1}(Y(s)\leq x)ds\,.$$

If $x<\inf_{s\in[\varepsilon,\varepsilon+(1-\alpha)T)}Y(s)$, we see that the first term vanishes, and since the second term is less than (or equal to) α, we must have $Q_Y\geq x$, and since x is arbitrary, we have

$$Q_Y\geq\inf_{s\in[\varepsilon,\varepsilon+(1-\alpha)T)}Y(s)\,,$$

and consequently

$$Q_Y\geq\sup_{\varepsilon\in(0,\alpha T]}\left\{\inf_{s\in[\varepsilon,\varepsilon+(1-\alpha)T)}Y(s)\right\},$$

and finally we have arrived at (2.1). $\square$

It is not obvious whether the two bounds in (2.1) in general are identical and therefore we have unfortunately not obtained

$$Q_Y=\inf_{\varepsilon\in[0,(1-\alpha)T)}\left\{\sup_{s\in(0,\alpha T]}Y(s+\varepsilon)\right\}.$$

However, if Y is linear, that is, $Y(t)=\beta\,t$, for some real-valued constant β, then we do have equality in (2.1). In fact, since $Y(s+\varepsilon)=Y(s)+Y(\varepsilon)$ (and Y is continuous), we obtain from (2.1) that

$$Q_Y=\sup_{s\in[0,\alpha T]}Y(s)+\inf_{s\in[0,(1-\alpha)T]}Y(s)\,.$$

Since processes with stationary and independent increments have paths that are 'linear in distribution', the following theorem will appear.

Theorem 2.2 *Let $X = (X(t))_{t\geq 0}$ be some real-valued stochastic process with paths that are right-continuous with left-hand limits, and assume that X has stationary and independent increments, and $X(0) = 0$. Then for $\alpha \in [0, 1)$, we have*

$$Q_X \stackrel{\mathrm{D}}{=} \sup_{s\in[0,\alpha T]} X(s) + \inf_{s\in[0,(1-\alpha)T]} \tilde{X}(s)\,, \tag{2.3}$$

where $\stackrel{\mathrm{D}}{=}$ stands for equality in distribution, and X and $\tilde{X}$ are independent copies.

Proof: Denote the left-hand and right-hand side in (2.1) by L_Y and R_Y, respectively. Since X has stationary and independent increments with $X(0) = 0$, the process $(X(s+\varepsilon) - X(\varepsilon))_{s\geq 0}$ is, for an arbitrary ε, independent of $X(\varepsilon)$, and has the same distribution as X. Therefore, we can write

$$X(s+\varepsilon) \stackrel{\mathrm{D}}{=} X(\varepsilon) + \tilde{X}(s) \tag{2.4}$$

where the process $\tilde{X} = (\tilde{X}(s))_{s\geq 0}$ has the same distribution as X and is independent of $X(\varepsilon)$. A fortiori, this holds if we take for $\tilde{X}$ an independent copy of X.

We shall also state a finite dimensional version of (2.4). Let $n > 1$ be a finite number, and $0 = s_0 + \varepsilon_0 < s_1 + \varepsilon_1 < \cdots < s_n + \varepsilon_n$, some (non-negative) values, where we without restriction further can assume $s_{j-1} < s_j$, and $\varepsilon_{j-1} < \varepsilon_j$, for $j = 1, \ldots, n$. Then for any 'small' interval $\mathrm{d}\eta_j$ around $\eta_j \in \mathcal{R}$, ($\mathrm{d}\eta_0 = 0$) we find together with (2.4)

$$\begin{aligned}
& P(X(s_1+\varepsilon_1) \in \mathrm{d}\eta_1\,, \ldots,\, X(s_n+\varepsilon_n) \in \mathrm{d}\eta_n) \\
= \;& P(X(s_j+\varepsilon_j) - X(s_{j-1}+\varepsilon_{j-1}) \in \mathrm{d}(\eta_j - \eta_{j-1})\,,\;\; j = 1, \ldots, n) \\
= \;& \prod_{j=1}^{n} P(X(s_j - s_{j-1} + \varepsilon_j - \varepsilon_{j-1}) \in \mathrm{d}(\eta_j - \eta_{j-1})) \\
= \;& \prod_{j=1}^{n} P(X(\varepsilon_j - \varepsilon_{j-1}) + \tilde{X}(s_j - s_{j-1}) \in \mathrm{d}(\eta_j - \eta_{j-1})) \\
= \;& P(X(\varepsilon_1) + \tilde{X}(s_1) \in \mathrm{d}\eta_1\,, \ldots,\, X(\varepsilon_n) + \tilde{X}(s_n) \in \mathrm{d}\eta_n)\,.
\end{aligned} \tag{2.5}$$

Since *inf* and *sup* associate measurable mappings on the space of functions that are right-continuous with left-hand limits, we get in particular by (2.5), that for any finite sets $F, G \subset [0, \infty)$

$$\sup_{\varepsilon\in G} \left\{ \inf_{s\in F} X(s+\varepsilon) \right\} \stackrel{\mathrm{D}}{=} \sup_{\varepsilon\in G} X(\varepsilon) + \inf_{s\in F} \tilde{X}(s)\,. \tag{2.6}$$

Let then $\mathcal{S}$ represent a dense subset of $\mathcal{R}$, for instance the rational numbers, and consider $F_m, G_n \subset \mathcal{S}$, $m, n = 1, 2, \ldots$, increasing sequences of finite sets, such that $\lim_m F_m = \mathcal{S}$, and $\lim_n G_n = \mathcal{S}$. By (2.6) we conclude that

$$\sup_{\varepsilon\in(0,\alpha T]\cap G_n} \left\{ \inf_{s\in[0,(1-\alpha)T)\cap F_m} X(s+\varepsilon) \right\} \stackrel{\mathrm{D}}{=} \sup_{\varepsilon\in(0,\alpha T]\cap G_n} X(\varepsilon) + \inf_{s\in[0,(1-\alpha)T)\cap F_m} \tilde{X}(s)\,. \tag{2.7}$$

Fixing first n, and letting m tend to infinity (and finally letting n tend to infinity) in (2.7), we can use the standard 'continuity property' of a probability measure for increasing (and decreasing) sets, to obtain

$$\sup_{\varepsilon\in(0,\alpha T]\cap \mathcal{S}} \left\{ \inf_{s\in[0,(1-\alpha)T)\cap \mathcal{S}} X(s+\varepsilon) \right\} \stackrel{\mathrm{D}}{=} \sup_{\varepsilon\in(0,\alpha T]\cap \mathcal{S}} X(\varepsilon) + \inf_{s\in[0,(1-\alpha)T)\cap \mathcal{S}} \tilde{X}(s)\,.$$

Thus

$$\mathrm{L}_X \stackrel{\mathrm{D}}{=} \sup_{\varepsilon\in[0,\alpha T]} X(\varepsilon) + \inf_{s\in[0,(1-\alpha)T]} \tilde{X}(s)\,,$$

where we overall due to the right-continuity can disregard $\mathcal{S}$, and also for the first term used that $\sup_{\varepsilon\in[0,\alpha T]} X(\varepsilon) = \sup_{\varepsilon\in(0,\alpha T]} X(\varepsilon)$ (right-continuity), and for the second term the fact that $\tilde{X}$ cannot have any fixed times of discontinuity (it has stationary and independent increments), implying that $\inf_{s\in[0,(1-\alpha)T]} \tilde{X}(s) = \inf_{s\in[0,(1-\alpha)T)} \tilde{X}(s)$ a.s. Repeating these argument starting with interchanging *sup* and *inf* on the left of (2.6), we obtain the same result for R_X, and consequently (2.3) is proved. $\square$

Identity (2.3) was formulated and proved in [1], mainly by using the result in [3] (Wendel), who used a characteristic function argument to obtain the result with the discrete time version of the process $W = (W(t))_{t\geq 0}$

$$W(t) = \sum_{k=1}^{[t]} Z_k\,, \qquad W(0) = 0\,,$$

where $Z_1, Z_2, \ldots$ are i.i.d. random variables, and $[t]$ is the integer value of t. So (2.3) could also be proved by making an appropriate discretiziation of $[0,T]$, and take $Z_k = X(k/n) - X((k-1)/n)$, say, for $k = 1, 2, \ldots, [Tn]$, and finally make a limit argument as n tends to infinity. This seems what the approach in [1] is aiming at.

With the technique presented for proving (2.3), we could also obtain

$$Q_W \stackrel{\mathrm{D}}{=} \sup_{s\in[0,\alpha T]} W(s) + \inf_{s\in[0,(1-\alpha)T)} \tilde{W}(s)\,, \tag{2.8}$$

which is the time continuous analogue of the classical result by Wendel, where W and $\tilde{W}$ are independent copies. We omit further comments and details, but shall instead focus on the more general *renewal reward process*, which is the jump process

$$R(t) = \sum_{k=1}^{N(t)} Y_k\,,$$

where the pairs $(S_1, Y_1), (S_2, Y_2), \ldots$ are i.i.d., and $S_n = T_n - T_{n-1}$, $n = 1, 2, \ldots$, are the inter-occurrence times where T_n is the time of the nth jump ($T_0 = 0$), and $N(t)$ is the number of jumps over $[0,t]$. We state the result in [2] in the following theorem, and consequently comment on a slight discrepancy to his version.

Theorem 2.3 *Let $R = (R(t))_{t\geq 0}$ and $\tilde{R} = (\tilde{R}(t))_{t\geq 0}$ be independent copies of a renewal reward process. Then*

$$Q_R \stackrel{\mathrm{D}}{=} \sup_{s\in[0,\alpha T]} R(s) + \inf_{s\in[0,(1-\alpha)T)} \tilde{R}(s)\,. \tag{2.9}$$

Proof: Since $R(t)$ only changes at the jump times, we can write

$$\inf_{t\in[0,\eta]} R(t) = \inf_{T_n\in[0,\eta]} R(T_n)\,, \quad \eta > 0\,,$$

(and similarly with *sup*). We then find that

$$\begin{aligned}
\sup_{\varepsilon\in(0,\alpha T]} \left\{ \inf_{s\in[0,(1-\alpha)T)} R(s+\varepsilon) \right\} \quad &= \quad \sup_{T_n\in[0,\alpha T]} \left\{ \inf_{T_{n+m}-T_n\in[0,(1-\alpha)T)} R(T_{n+m}) \right\} \\
&\stackrel{\mathrm{D}}{=} \quad \sup_{T_n\in[0,\alpha T]} R(T_n) + \inf_{\tilde{T}_m\in[0,(1-\alpha)T)} \tilde{R}(\tilde{T}_m) \\
&= \quad \sup_{s\in[0,\alpha T]} R(s) + \inf_{s\in[0,(1-\alpha)T)} \tilde{R}(s)\,,
\end{aligned}$$

where $\tilde{T}_1, \tilde{T}_2, \ldots$ are the jump times of $\tilde{R}(t)$, and we used that

$$R(T_{n+m}) \stackrel{\mathrm{D}}{=} R(T_n) + \tilde{R}(\tilde{T}_m)\,,$$

and as in proof of Theorem 2.2, we can make any finite dimension distribution identity. Consequently, these observations lead to (2.9). $\square$

The approach in [2] is mainly basd on some integral equation techniques (that seem specific for the case) for his prove of (2.9), which he states with the second term as $\inf_{s\in[0,(1-\alpha)T]}\tilde{R}(s)$, and can make a difference to (2.9) if fixed non-stochastic times of jumps are considered (take e.g. all $(S_n, Y_n) = (1, -1)$, $T = 2$ and $\alpha = 1/2$, and calculate Q_R).

We shall finally consider a simple example, where the process has independent but in general no version with stationary increments, and then see how the bounds in (2.1) still can give some result.

Example 2.1. Consider the Gaussian process $X = (X(t))_{t\geq 0}$

$$X(t) = B(t) + \int_0^t \lambda(s)\mathrm{d}s\,,$$

where $B = (B(t))_{t\geq 0}$ is a standard Brownian motion, and $\lambda(t)$ is a piecewise continuous function. Below, *increasing* means *non-decreasing* and vice versa. Assume first that $\lambda(t)$ is increasing, and with $\Lambda(t) = \int_0^t \lambda(s)\mathrm{d}s$, we then have

$$\Lambda(s+t) \geq \Lambda(s) + \Lambda(t)\,, \qquad s,t\geq 0\,. \tag{2.10}$$

With the same notation as in Theorem 2.2, and since a Brownian motion has stationary and independent increments, we get together with (2.10) that

$$X(s+\varepsilon) \overset{\mathrm{D}}{\geq} X(\varepsilon) + \tilde{X}(s)\,, \tag{2.11}$$

which means in terms of distribution functions, that

$$P(X(s+\varepsilon)\leq u) \leq P(X(\varepsilon)+\tilde{X}(s)\leq u)\,, \qquad u\in\mathcal{R}\,.$$

To obtain this inequality in finite dimension, let $0 = s_0+\varepsilon_0 < s_1+\varepsilon_1 < \cdots < s_n+\varepsilon_n$, be as in the proof of Theorem 2.2, and by (2.5) we immediately get

$$\begin{aligned} & P(B(s_1+\varepsilon_1)\leq u_1\,,\ldots,\, B(s_n+\varepsilon_n)\leq u_n) \\ = \quad & P(B(\varepsilon_1)+\tilde{B}(s_1)\leq u_1\,,\ldots,\, B(\varepsilon_n)+\tilde{B}(s_n)\leq u_n)\,, \end{aligned} \tag{2.12}$$

where $u_1,\ldots,u_n\in\mathcal{R}$, and we recall $\tilde{B}(t) = \tilde{X}(t) - \Lambda(t)$. Using first (2.10) and then (2.12), we get

$$\begin{aligned} & P(X(s_1+\varepsilon_1)-\Lambda(s_1)-\Lambda(\varepsilon_1)\leq u_1\,,\ldots,\, X(s_n+\varepsilon_n)-\Lambda(s_n)-\Lambda(\varepsilon_n)\leq u_n) \\ \leq \quad & P(B(s_1+\varepsilon_1)\leq u_1\,,\ldots,\, B(s_n+\varepsilon_n)\leq u_n) \\ = \quad & P(X(\varepsilon_1)+\tilde{X}(s_1)-\Lambda(s_1)-\Lambda(\varepsilon_1)\leq u_1\,,\ldots,\, X(\varepsilon_n)+\tilde{X}(s_n)-\Lambda(s_n)-\Lambda(\varepsilon_n)\leq u_n)\,, \end{aligned}$$

and since this holds for any $u_1,\ldots,u_n$, we conclude that

$$\begin{aligned} & P(X(s_1+\varepsilon_1)\leq u_1\,,\ldots,\, X(s_n+\varepsilon_n)\leq u_n) \\ \leq \quad & P(X(\varepsilon_1)+\tilde{X}(s_1)\leq u_1\,,\ldots,\, X(\varepsilon_n)+\tilde{X}(s_n)\leq u_n)\,, \end{aligned}$$

for any $u_1,\ldots,u_n\in\mathcal{R}$, which states our finite dimensional version of (2.11). Consequently, for any finite sets $F, G\subset[0,\infty)$, we have

$$\sup_{\varepsilon\in G}\left\{\inf_{s\in F} X(s+\varepsilon)\right\} \overset{\mathrm{D}}{\geq} \sup_{\varepsilon\in G} X(\varepsilon) + \inf_{s\in F}\tilde{X}(s)\,,$$

and repeating the steps following (2.6), we find

$$\mathrm{L}_X \overset{\mathrm{D}}{\geq} \sup_{s\in[0,\alpha T]} X(s) + \inf_{s\in[0,(1-\alpha)T]}\tilde{X}(s)\,,$$

and therefore

$$Q_X \overset{\mathrm{D}}{\geq} \sup_{s\in[0,\alpha T]} X(s) + \inf_{s\in[0,(1-\alpha)T]} \tilde{X}(s)\,.$$

In the same manner we find that

$$Q_X \overset{\mathrm{D}}{\leq} \sup_{s\in[0,\alpha T]} X(s) + \inf_{s\in[0,(1-\alpha)T]} \tilde{X}(s)\,,$$

if $\lambda(t)$ is decreasing. $\square$

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