Aspects of Prospective Mean Values in Risk Theory

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Abstract

The present paper deals with conditional mean values for analysing prospective events in risk theory, mainly related to reserve evaluation. In some (Markov) cases, for instance the classical life insurance set-up, Kolmogorov's backward differential equations suffice as a constructive tool, together with basic martingale relations. However, in many important (Markov) cases we need more refined martingale techniques. We shall mainly focus on cases with random time horizon defined as an exit time. The martingale results are carried out in a marked point process set-up, hereunder by use of the important concept of an intensity measure.

Keywords: Thiele's differential equation, compound distribution, martingale, optional sampling, marked point process, exit time.

1 Introduction

Applications of stochastic calculus in life and non-life insurance have until now not been a major issue. It has been lightly touched in life insurance where the classical idea is to model the policy in accordance with a Markov (jump) process of finite state space. Using the nice structure of Kolmogorov's backward differential equations, one can establish the so-called Thiele's differential equations, which give a tool for identifying the prospective reserves, see Hoem (1969). However, he did not use any kind of martingale relations.

A generalization of the classical life insurance model was studied by Møller (1993) in a semi-Markov case where payment functions and transition intensities were allowed to depend on the duration in the visiting state. A simple martingale argument seemed sufficient to obtain the result. We shall here combine the methods in Møller (1993) and (1995) to see how conditional expectations can be obtained and evaluated under a random time horizon, which in particular will be defined as an exit time for some measurable set. We give a practical example.

Basic in the analysis is that our stochastic phenomena occur at random times (points), $T_1 < T_2, \ldots$, and are represented by corresponding marks $Z_1, Z_2, \ldots$. The sequence $(T_n, Z_n)_{n \geq 1}$ is called a marked point process, see below, and in this manner our formulation is applicable for both life and non-life insurance problems, and many other situations of interest in risk theory.

Davis (1993), who introduced the concept of PD (piecewise-deterministic) Markov processes, has also shown interest for conditional means, but his motivation is different (extended generator, strong Markov property, cemetary state).

In Section 2 we outline the concept of a marked point process and the associated martingale results as stated in Brémaud (1981).

In Section 3, we stress the martingale approach and the techniques to obtain prospective mean values from a system of differential equations. First we pay attention to the semi-Markov model in Møller (1993). We shall present a different proof based on Doob's optional sampling theorem. Then we mention the classical Markov jump case under a fixed period of time. Finally, we shall indicate how a compound distribution function can be obtained, which is also analysed in Møller (1996) for fixed finite time horizons. Throughout we assume that the force of interest is fixed over time. However generalizations to stochastic interest are indeed feasible.

2 Some elements of point process theory

Assume there is given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ satisfying the usual conditions, that is, the space is complete and the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous. In the sequel all random variables are assumed to be defined on $(\Omega, \mathcal{F}, P)$.

Basic in the studies are the concept of marked point processes and the associated martingale theory. A point process is a sequence $(T_n, Z_n)_{n\geq 1}$ of stochastic pairs, where $T_1, T_2, \ldots$ are non-negative and represent times of occurrence of some phenomena represented by the stochastic elements $Z_1, Z_2, \ldots$, called the marks, which are assumed to take values in some measurable space $\mathcal{Z}$ endowed with a σ-algebra $\mathcal{E}$. This model framework can be convenient for the analysis of insurance problems. In non-life insurance, the points typically represent times of occurrence of claims, and the marks represent the individual claim amounts. In life insurance the points could represent times of transition between states of a process governing the policy, and the state entered or the pair of states involved in the transition could represent the mark.

Let $\mathcal{R}$ and $\mathcal{R}_+$ denote the real line and non-negative half line endowed with their usual Borel σ-algebras $\mathcal{B}$ and $\mathcal{B}_+$, respectively. Let $I(F)$ denote the indicator of a set F in $\mathcal{F}$. Introduce for each $A \in \mathcal{E}$ the counting measure

$$N(t, A) = \sum_{i=1}^{\infty} I(T_i \leq t,\, Z_i \in A), \tag{2.1}$$

which counts the number of jumps in the time interval $(0, t]$ with marks taking values in A. In particular $N(t) = N(t, \mathcal{Z})$. The counting processes lead to the natural filtration

$$\mathcal{F}_t^N = \sigma(N(s, A),\, s \leq t,\, A \in \mathcal{E}).$$

Another important issue is the existence of an intensity process: Assume that $N(t, A)$ admits an $\mathcal{F}_t$-intensity $\nu(t, A)$ ($\mathcal{F}_t^N \subset \mathcal{F}_t$) assumed to be bounded over finite intervals, informally defined as

$$\nu(t, A)dt = E(N(dt, A) \,|\, \mathcal{F}_{t-}) + o(dt), \tag{2.2}$$

where $\mathcal{F}_{t-} = \vee_{s<t}\mathcal{F}_s$ is the information prior to time t. We abbreviate $\nu(t) = \nu(t, \mathcal{Z})$, which is the intensity of $N(t)$. We can also write the intensity on the form

$$\nu(t, A) = \nu(t)G(t, A), \quad G(t, A) = \int_{z\in A} G(t, dz), \tag{2.3}$$

where $G(t, A)$ is a probability, and is interpreted as the conditional probability given all information prior to time t and that a jump occurred at time t, that the associated mark will belong to A. An important result (e.g. Brémaud, 1981, pp. $27, 235$) states that the process

$$M_t = \int_{(0,t]} \int_{z\in\mathcal{Z}} H(s, z)(N(ds, dz) - \nu(s, dz)ds),$$

where H is some $\mathcal{F}_t$-predictable process (indexed by $\mathcal{Z}$) is a zero mean $\mathcal{F}_t$-martingale, that is,

$$E\left[\int_{(t,v]} \int_{z\in\mathcal{Z}} H(s, z)N(ds, dz)\middle|\mathcal{F}_t\right] = E\left[\int_{(t,v]} \int_{z\in\mathcal{Z}} H(s, z)\nu(s, dz)ds\middle|\mathcal{F}_t\right], \tag{2.4}$$

for any $t < v$, whenever

$$E\left[\int_{(0,t]} \int_{z\in\mathcal{Z}} |H(s, z)|\nu(s, dz)ds\right] < \infty,$$

for any $t > 0$. In the sequel it should be sufficient to know that, in particular, any process with left continuous or deterministic paths (indexed by $\mathcal{Z}$) is predictable.

In the sequel we will also write $\int_s^t$ and $\int_A$ instead of $\int_{(s,t]}$ and $\int_{z\in A}$, respectively.

3 The martingale approach

Let $X = (X_t)_{t\geq 0}$ be a jump process taking values in the measurable space $(\mathcal{Z}, \mathcal{E})$. This leads to the associated marked point process $(T_n, Z_n)_{n\geq 1}$, where $T_1 < T_2 < \dots$ denote the jump times of X, and $Z_1, Z_2, \dots$ are the corresponding values of X at the jumps, $Z_n = X_{T_n}$. Define the cadlag (right continuous with left hand-limits) process

$$U_t = \sum_{n\geq 0} (t - T_n) I(T_n \leq t < T_{n+1}), \tag{3.1}$$

that measures the time spent in the visiting state of X_t. We shall assume that the process $\tilde{X} = (\tilde{X}_t)_{t\geq 0}$, $\tilde{X}_t = (X_t, U_t)$, is a (PD) Markov process given by transition functions

$$p(s, \tilde{x};\, t, B) = P(\tilde{X}_t \in B \,|\, \tilde{X}_s = \tilde{x}), \quad s < t,$$

where B is in the σ-algebra $\mathcal{E} \otimes \mathcal{B}_+$ and $\tilde{x} = (x, u)$. Further, we assume that $\tilde{X}$ admits an intensity function $(t, u, x) \to \lambda_{(x,u)}(t, A)$ for each $A \in \mathcal{E}$, given by

$$\lambda_{(x,u)}(t, A) = \lim_{h \searrow 0} \frac{p(t, \tilde{x};\, t + h, A \times \mathcal{R}_+)}{h}, \quad x \notin A.$$

Again we write $\lambda_{(x,u)}(t) = \lambda_{(x,u)}(t, \mathcal{Z}\backslash x)$. Then the intensity process of $N(t, A)$ exists and is given by

$$\nu(t, A) = \lambda_{\tilde{X}_t}(t, A) I(X_t \notin A).$$

We introduce the measurable payment functions $(t, u, x) \to \zeta_{(x,u)}(t)$ and $(t, u, x, y) \to b_{(x,u)}(t, y)$, where $\zeta_{(x,u)}(t)$ plays the role of an (annual) annuity or premium rate, assumed to be non-negative (negative) if it represents premium (annuity, pension). If X jumps from x to y at time t with a duration of u in state x, an amount of $b_{(x,u)}(t, y)$ is paid to the insured. We assume that the annual interest is fixed and let δ denote the corresponding force of interest.

Let now our stochastic risk business be represented by a stochastic process $(R_t)_{t\geq 0}$ driven by the (stochastic) differential equation

$$dR_t = \delta dt\, R_t + \zeta_{\tilde{X}_t}(t)dt - \int_{\mathcal{Z}} b_{\tilde{X}_{t-}}(t, z) N(dt, dz), \tag{3.2}$$

which has a self-explained interpretation. The minus in (3.2) indicates that $b_{(x,u)}(t, y)$ is normally interpreted as non-negative. Let first $T \leq \infty$, be a fixed period of time. Using integration by parts, it is readily checked that the process

$$Q_t(T) = \int_t^T e^{-\delta(s-t)} \left[\int_{\mathcal{Z}} b_{\tilde{X}_{s-}}(s, z) N(ds, dz) - \zeta_{\tilde{X}_s}(s) ds \right] + l_{\tilde{X}_T}(T) e^{-\delta(T-t)}, \tag{3.3}$$

satisfies (3.2) with $R_T = l_{\tilde{X}_T}(T)$, where $(t, u, x) \to l_{(x,u)}(t)$ is some measurable function representing a cost by time T. We shall use the convention $l_{\tilde{X}_T}(T) e^{-\delta T} = 0$ if $T = \infty$, obtained for instance by replacing $l_{(x,u)}(t)$ with $l_{(x,u)}(t) I(t < \infty)$.

An actuary would then typically be interested in the conditional mean $E[Q_t(T) \,|\, \mathcal{F}_t^N]$ for $t \in [0, T]$, which due to Markov property only depends on $\mathcal{F}_t^N$ via $\tilde{X}_t$, and together with (2.4) the expectation is given by

$$\begin{aligned} E[Q_t(T) \,|\, \tilde{X}_t = (x, u)] \quad = \quad & \int_t^T e^{-\delta(s-t)} E_{(t,x,u)} \left[\int_{\mathcal{Z}\backslash X_s} b_{\tilde{X}_s}(s, z) \lambda_{\tilde{X}_s}(s, dz) - \zeta_{\tilde{X}_s}(s) \right] ds \\ & + e^{-\delta(T-t)} E_{(t,u,x)} \left[\, l_{\tilde{X}_T}(T) \,\right], \end{aligned} \tag{3.4}$$

where $E_{(t,u,x)}$ denotes the conditional expectation given $\tilde{X}_t = (x, u)$. If we knew a version of 'Kolmogorov's backward differential equations' for $\tilde{X}$ we could simply evaluate $E_{(t,u,x)}$ wrt. these, and differentiate to obtain a system of differential equations for (3.4) of similar structure. So this approach can be done if X is of Markov type, implying that $\lambda_{(x,u)}(t)$ is assumed independent of u, and if furthermore all payment functions are not depending on u, see comments following (3.10). For classical Markov theory we refer to Doob (1953).

To open for further flexibility and applications, we shall complicate matters further by operating under a random time horizon given by some $\mathcal{F}_t$-stopping time τ, that is, $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. In particular we shall assume that τ is given as the first exit time

$$\tau_{BC} = \inf\{t \geq 0 : \tilde{X}_t \notin BC\},$$

for some set $BC = B \times C$, where $B \in \mathcal{E}$ and $C \in \mathcal{B}_+$. Define also

$$\begin{aligned}\tau_{BC,t} &= \inf\{s \geq t : \tilde{X}_s \notin BC\} \\ &= t + \inf\{s \geq 0 : \tilde{X}_{t+s} \notin BC\}.\end{aligned}$$

Note that $\tau_{BC,\tau_{BC}} = \tau_{BC}$. We define $\tau_{BC,t} = \infty$ for any t, if $\tilde{X}_s \in BC$ for all $s \geq t$. We write

$$Q_0(\tau_{BC}) = \int_0^{\tau_{BC}} e^{-\delta s}\left[\int_{\mathcal{Z}} b_{\tilde{X}_{s-}}(s,z)N(ds,dz) - \zeta_{\tilde{X}_s}(s)ds\right] + l_{\tilde{X}_{\tau_{BC}}}(\tau_{BC})e^{-\delta\tau_{BC}},$$

and define the process

$$\Gamma_t = \int_0^t e^{-\delta s}\left[\int_{\mathcal{Z}} b_{\tilde{X}_{s-}}(s,z)N(ds,dz) - \zeta_{\tilde{X}_s}(s)ds\right].$$

Then for each $t > 0$, we can decompose as

$$\begin{aligned}Q_0(\tau_{BC}) &= I(\tau_{BC} \leq t)Q_0(\tau_{BC}) + I(\tau_{BC} > t)\{\Gamma_t + Q_t(\tau_{BC,t})e^{-\delta t}\} \\ &= \Gamma_{\tau_{BC}\wedge t} + I(\tau_{BC} \leq t)l_{\tilde{X}_{\tau_{BC}}}(\tau_{BC})e^{-\delta\tau_{BC}} + I(\tau_{BC} > t)\, Q_t(\tau_{BC,t})e^{-\delta t},\end{aligned} \tag{3.5}$$

where we have used that

$$I(\tau_{BC} > s) = I(\tau_{BC} > t)I(\tau_{BC,t} > s), \quad \forall s > t,$$

and we recall from (3.3) that

$$Q_t(\tau_{BC,t}) = \int_t^{\tau_{BC,t}} e^{-\delta(s-t)}\left[\int_{\mathcal{Z}} b_{\tilde{X}_{s-}}(s,z)N(ds,dz) - \zeta_{\tilde{X}_s}(s)ds\right] + l_{\tilde{X}_{\tau_{BC,t}}}(\tau_{BC,t})e^{-\delta(\tau_{BC,t}-t)}.$$

We are now concerned with evaluating the mean process

$$V_{\mathcal{F}_t^N}(t) = E[Q_t(\tau_{BC,t})\,|\,\mathcal{F}_t^N], \quad t \geq 0,$$

which again due to the $\mathcal{F}_t^N$-Markov property depends through $\mathcal{F}_t^N$ only via $\tilde{X}_t$, which we abbreviate $V_{\tilde{X}_t}(t)$, where $(t,u,x) \to V_{(x,u)}(t)$ is the measurable function

$$V_{(x,u)}(t) = E[Q_t(\tau_{BC,t})\,|\,\tilde{X}_t = (x,u)].$$

We now have the boundary condition

$$V_{(x,u)}(t) = l_{(x,u)}(t), \quad \forall (x,u) \notin BC. \tag{3.6}$$

For identifying $(t,u) \to V_{(x,u)}(t)$ (for $(x,u) \in BC$) from a system of differential equations we first assume that $E|Q_0(\tau_{BC})| < \infty$, and define the $\mathcal{F}_t^N$-martingale

$$M_t = E[Q_0(\tau_{BC})\,|\,\mathcal{F}_t^N], \quad t \geq 0\,.$$

By (3.5) and (3.6) we can write

$$
\begin{aligned}
M_t &= \Gamma_{\tau_{BC}\wedge t} + I(\tau_{BC} \le t) l_{\tilde{X}_{\tau_{BC}}}(\tau_{BC})\, e^{-\delta\tau_{BC}} + I(\tau_{BC} > t)\, e^{-\delta t}\, V_{\tilde{X}_t}(t) \\
&= \Gamma_{\tau_{BC}\wedge t} + e^{-\delta\,\tau_{BC}\wedge t}\, V_{\tilde{X}_{\tau_{BC}\wedge t}}(\tau_{BC}\wedge t), \quad t \ge 0. \qquad (3.7)
\end{aligned}
$$

This relation can be used to prove the following generalization of Theorem 3.1 in Møller (1993).

Theorem 3.1 *Let $t \to D_t$ be a function playing the role of U_t between the jumps, that is, D_t is non-negative with derivative $dD_t = dt$. Then over the continuity points of $\lambda_{(x,u)}(t, A)$, $b_{(x,u)}(t, z)$ and $\zeta_{(x,u)}(t)$, the functions $t \to V_{(x,D_t)}(t)$ satisfy the system of differential equations*

$$
\begin{aligned}
\frac{dV_{(x,D_t)}(t)}{dt} &= [\delta + \lambda_{(x,D_t)}(t)]\, V_{(x,D_t)}(t) + \zeta_{(x,D_t)}(t) \\
&\quad - \int_{\mathcal{Z}\setminus x} \lambda_{(x,D_t)}(t, dz)\, [b_{(x,D_t)}(t, z) + V_{(z,0)}(t)], \quad t \in (0, \tau_C^*), \;\; x \in B, \qquad (3.8)
\end{aligned}
$$

with the boundary condition

$$
V_{(x,u)}(t) = l_{(x,u)}(t), \quad \forall (x, u) \notin BC, \;\; t \ge 0,
$$

where τ_C^* is the (deterministic) time

$$
\tau_C^* = \inf\{t \ge 0 : \, D_t \notin C\}.
$$

Proof: Stopping the martingale at T_1 the first jump time of X, the process $t \to M_{t\wedge T_1}$ becomes an $\mathcal{F}_t^N$-martingale (optional sampling), and then in particular has the constant mean value

$$
E_{(x,u)}[M_{t\wedge T_1}] = V_{(x,u)}(0), \quad \forall t \ge 0.
$$

However, we observe that $\tau_{BC} \wedge T_1 = \tau_C^* \wedge T_1$, $P_{(x,u)}$-a.s. for any $(x, u) \in BC$, since we can only leave BC (via C) deterministically before T_1 when we start in BC, and if do not leave BC before T_1, then this will in particular not occur deterministically. Using (3.7), we then get for any $t \in (0, \tau_C^*)$

$$
\begin{aligned}
M_{t\wedge T_1} &= \Gamma_{t\wedge T_1} + e^{-\delta\, t\wedge T_1}\, V_{X_{t\wedge T_1}}(t \wedge T_1) \\
&= \Gamma_{t\wedge T_1} + I(T_1 > t) e^{-\delta t} V_{(x,u+t)}(t) + I(T_1 \le t) e^{-\delta T_1} V_{(Z_1,0)}(T_1). \qquad (3.9)
\end{aligned}
$$

The $P_{(x,u)}$-distribution of (T_1, Z_1) is given by

$$
P_{(x,u)}(T_1 \in dt,\, Z_1 \in A) = \exp\left(-\int_0^t \lambda_{(x,u+s)}(s) ds\right) \lambda_{(x,u+t)}(t, A) dt.
$$

For the first term in (3.9), we can by (2.4) write

$$
\begin{aligned}
E_{(x,u)}[\Gamma_{t\wedge T_1}] &= E_{(x,u)} \int_0^t I(T_1 \ge s) e^{-\delta s} \left[\int_{\mathcal{Z}\setminus x} b_{(x,u+s)}(s, z) \lambda_{(x,u+s)}(s, dz) - \zeta_{(x,u+s)}(s)\right] ds \\
&= \int_0^t e^{-\int_0^s (\delta + \lambda_{(x,u+\eta)}(\eta)) d\eta} \\
&\qquad \left[\int_{\mathcal{Z}\setminus x} b_{(x,u+s)}(s, z) \lambda_{(x,u+s)}(s, dz) - \zeta_{(x,u+s)}(s)\right] ds
\end{aligned}
$$

and by evaluating the rest of the (constant) mean in (3.9), we can finally, by differentiation, arrive at (3.8) with $D_t = u + t$, and since this only depends on t, u via the sum, we get the desired result. $\square$

Note that (3.8) can be viewed as a system of partial differential equations. Namely, if we assume that $(t, u) \to V_{(x,u)}(t)$ has continuous partial derivatives for $t \in (0, \tau_C^*)$ and $u > 0$, we can write

$$\frac{dV_{(x,D_t)}(t)}{dt} = \frac{\partial V_{(x,D_t)}(t)}{\partial t} + \frac{\partial V_{(x,D_t)}(t)}{\partial u}, \quad t \in (0, \tau_C^*).$$

In the case where τ_{BC} is bounded by a finite horizon T, say, occurring e.g. if τ_{BC} is replaced by $\tau_{BC} \wedge T$, we can then hope to evalute (3.8) (numerically) together with the initial condition

$$V_{(x,u)}(T) = l_{(x,u)}(T), \quad \forall (x, u) \in BC.$$

A special case that leads to studying the distribution of first exit time for $\tilde{X}$, is obtained with $b_{(x,u)}(t, y)$, $\zeta_{(x,u)}(t)$ and δ identically equal to zero, and furthermore with $l_{(x,u)}(t) = I(t < T)$ for some $T \leq \infty$. Then

$$Q_t(\tau_{BC,t}) = I(\inf_{t \leq s < T} \tilde{X}_s \notin BC),$$

and

$$V_{(x,u)}(t) = P(\inf_{t \leq s < T} \tilde{X}_s \notin BC \,|\, \tilde{X}_t = (x, u)),$$

is the probability that $\tilde{X}$ exits BC after time t, given $\tilde{X}_t = (x, u)$. Also, the boundary condition obviously reads

$$V_{(x,u)}(t) = 1, \;\; (x, u) \notin BC.$$

We refer to Møller (1995) for some aspects of this probability for studying ruin probabilities for some PD-Markov processes.

Another special case is when X becomes Markov, obtained by letting $\lambda_{(x,u)}(t)$ be independent of u, which we denote by $\lambda_x(t)$. We then modify τ_{BC} as

$$\tau_D = \inf\{t \geq 0 : \, X_t \notin D\}, \;\; D \in \mathcal{E},$$

and $\tau_{D,t}$ accordingly. If we further let the payment functions be independent of u, denoted $\zeta_x(t)$, $b_x(t, y)$ and $l_x(t)$ respectively, and modify $Q_t(\;\;)$ to this case, the mean process

$$V_{\mathcal{F}_t^N}(t) = E[Q_t(\tau_{D,t}) \,|\, \mathcal{F}_t^N], \quad t \geq 0,$$

depends then on $\mathcal{F}_t^N$ only via X_t, and is given by the function $t \to V_x(t)$, defined as

$$V_x(t) = E[Q_t(\tau_{D,t}) \,|\, X_t = x].$$

Then we immediately obtain the following corollary.

Corollary 3.2 *Over the continuity points of $\lambda_x(t, A)$, $b_x(t, z)$ and $\zeta_x(t)$, the functions $t \to V_x(t)$ satisfy the system of differential equations*

$$\begin{aligned} \frac{dV_x(t)}{dt} &= [\delta + \lambda_x(t)]\, V_x(t) + \zeta_x(t) \\ &\quad - \int_{\mathcal{Z} \setminus x} \lambda_x(t, dz)\, [b_x(t, z) + V_z(t)], \quad t > 0, \;\; x \in D, \end{aligned} \tag{3.10}$$

with the boundary condition

$$V_x(t) = l_x(t), \quad \forall x \notin D, \;\; t \geq 0.$$

If we restrict to operate under a fixed period of time $T \leq \infty$, the system in (3.10) will hold for all $t \in (0,T)$ and $x \in \mathcal{Z}$ with no boundary condition. This is the trivial case and is merely a consequence of the classical Kolmogorov backward differential equations, which read

$$\frac{\partial p(s,x;\, t,A)}{\partial s} = \lambda_x(s)p(s,x;\, t,A) - \int_{\mathcal{Z}\setminus x} p(s,z;\, t,A)\lambda_x(s,dz),$$

where

$$p(s,x;\; t,A) = P(X_t \in A \,|\, X_s = x),$$

is the transition function. Modifying (3.4) accordingly, we can namely write

$$\begin{aligned} V_x(t) &= \int_t^T e^{-\delta(s-t)} \int_{\mathcal{Z}} \left[\int_{\mathcal{Z}\setminus w} b_w(s,z)\lambda_w(s,dz) - \zeta_w(s)ds \right] p(t,x;\; s,dw)ds \\ &\quad + e^{-\delta(T-t)} \int_{\mathcal{Z}} l_z(T)p(t,x;\, T,dz), \end{aligned}$$

and by differentiation the desired system for $V_x(t)$ arises.

An application of the results above could be:

Example 3.1. Let X be of Markov type given by the particular form $X_t = (S_t, N_t)$, where S_t has finite state space $\mathcal{J} = \{1,2,\ldots,J\}$ with J an integer, and N_t denotes the number of jumps of X over $[0,t]$. Let m be some fixed integer. In the following we interpret N_t as the number of claims over $[0,t]$.

Say that the company has the rule, that it as maximum tolerates m claims, and if the $m+1$th claim occurs at time t it immediately terminates the contract and thereafter pays an amount of $A(t)$ to the policyholder. This corresponds to a case with the random insurance period of

$$\tau = \inf\{t \geq 0 :\, N_t = m+1\},$$

which is equal to τ_D with

$$D = \mathcal{J} \times \{0,\ldots,m\}.$$

Since N_t only takes jumps af size 1, we obtain the reserves from (3.10) with

$$l_{(i,n)}(t) = A(t)I(n = m+1),$$

which corresponds to evaluate the system

$$\begin{aligned} \frac{dV_{(j,n)}(t)}{dt} &= [\delta + \lambda_{(j,n)}(t)]\, V_{(j,n)}(t) + \zeta_{(j,n)}(t) \\ &\quad - \sum_{i\neq j} \lambda_{(j,n)}(t;\, i,n+1)\, [b_{(j,n)}(t;\, i,n+1) + V_{(i,n+1)}(t)], \quad j \in \mathcal{J}, \quad n = 0,1\ldots, \end{aligned}$$

under the boundary conditions

$$V_{(i,m+1)}(t) = A(t), \quad V_{(i,n)}(t) = 0, \quad \forall i \in \mathcal{J}, \quad n = m+2,\ldots,$$

where $\lambda_{(j,n)}(t;\, i,n+1)$ is the intensity for the transition $(j,n) \to (i,n+1)$ at time t. So we have a finite (and then solvable) system. A reduction to a finite system will not appear if we had operate under a fixed period of time. Also, contrary to a fixed period of time, it is important to note that $V_{(i,n)}(t)$ still depends on n for any $i \in \mathcal{J}$ even if the intensity and payment functions are assumed independent of n. $\square$

There are of course many interesting cases to be studied using similar techniques and the necessary aspects of general martingale theory. For instance, another aspect of conditional mean could be to study the distribution of the random variable C_{τ_D}, where C_t is the jump process

$$\begin{aligned} C_t &= \sum_{i=1}^{N(t)} b_{X_{T_i-}}(T_i, Z_i) \\ &= \int_0^t \int_{\mathcal{Z}} b_{X_{s-}}(s,z) N(ds,dz). \end{aligned}$$

Define then the function $t \to F_x(t,r)$, $r \in \mathcal{R}$, as

$$F_x(t,r) = P\left(\int_t^{\tau_{D,t}} \int_{\mathcal{Z}} b_{X_{s-}}(s,z) N(ds,dz) \leq r \middle| X_t = x \right),$$

which satisfies the boundary condition

$$F_x(t,r) = I(r \geq 0), \quad \forall x \notin D.$$

Repeating the technique in (3.5) now with $Q_0(\tau_D) = I(\sum_{i=1}^{N(\tau_D)} b_{X_{T_i-}}(T_i, Z_i) \leq r)$, it will appear that the process

$$M_t = F_{X_{t\wedge\tau_D}}(t \wedge \tau_D, r - C_{t\wedge\tau_D}),$$

becomes an $\mathcal{F}_t^N$-martingale. Copying the same techniques for proving Theorem 3.1, we can arrive at the following theorem.

Theorem 3.3 *Over the continuity points of* $\lambda_x(t,A)$ *and* $b_x(t,z)$*, the functions* $t \to F_x(t,r)$ *satisfy for each* $r \in \mathcal{R}$*, the system of differential equations*

$$\frac{dF_x(t,r)}{dt} = \lambda_x(t)\, F_x(t,r) - \int_{\mathcal{Z}\setminus x} \lambda_x(t,dz)\, F_z(t, r - b_x(t,z)), \quad t > 0, \;\; x \in D, \tag{3.11}$$

with the boundary condition

$$F_x(t,r) = I(r \geq 0), \quad \forall x \notin D, \;\; t \geq 0.$$

As above we can evaluate (3.11) numerically if we replace τ_D with $\tau_D \wedge T$ for some finite T, and use the initial condition $F_x(T,r) = I(r \geq 0)$, for all $x \in D$.

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