A counting process approach to stochastic interest

CHRISTIAN MØLLER
BIOQUENCE, Annettevej 20, 2920 Charlottenlund, Denmark
chrm@bioquence.dk

Abstract

The aim of the present paper is to propose a stochastic approach for describing the return of an investment, and study its applications in insurance. The process governing the return of the investment is assumed to have bounded variation over finite intervals and possess a jump part. Attention is restricted to cases where the process has independent increments and is subject to fluctuations given by a Markovian environment. In the first case direct calculations are obtainable for evaluating moments of present and accumulated values. In the last case we establish differential equations akin to the celebrated Thiele's differential equation in life insurance.

Keywords: Doléans equation, Marked point process, Markovian environment, Payment functions.

1 Introduction

Pricing of insurance products is usually evaluated on a basis where no interest is taking into account or (in life insurance) is assumed to be fixed over time. On the other hand, the return of the investments made by the company (typically in shares and bonds) are affected over time by market values of the assets and inflation, the latter especially for long term investments in bonds. To obtain a more realistic assessment of the insurance companys solvency and the pricing of its products, it would benefit if the impact of the investment returns could be taken into account in the insurance benefits and evaluation basis (see e.g. Daykin et al. (1994) for some practical actuarial problems related to inflation and investment). In this paper we will suggest some ideas and techniques on such matters using the theory of marked point processes. We shall primarily focus on evaluating expected values for moments of present (discounted) and accumulated values. Other aspects on applications of point processes in risk theory can be seen in e.g. Møller (1991), (1993) and (1995).

Stochastic calculus applied in financial economics today is primarily used for pricing options and futures, and typically approached by modelling the return of a risky asset by some diffusion process. Extensions incorporating a jump part in the return process can e.g. be seen in Aase (1988) and references therein. Other motivations to stochastic interest can be found in Dufresne (1990), Paulsen (1993), Dietz (1992) and Bühlmann (1992).

In Section 2, we outline the basic idea and introduce the concept of a marked point process and its associated intensity measure, which are the building stones for further applications. In Subsection 2.1 we assume that the interest process has independent increments, and we make direct calculations for evaluating higher order moments of discounted and accumulated values. In Subsection 2.2, we treat a more complex case where the economy is assumed to be heterogeneous modelled in accordance with a Markovian environment represented by a finite number of states. For this model we can at least obtain a system of differential equations for the respective mean values.

In Section 3, we extend the model from Subsection 2.2 by incorporating payment functions, depending on the different states of the environment, which typically should play the role of e.g. premium, pensions and lump sum payments. We shall there focus on the reserve process, which is defined as the conditional expected value of future discounted net expenses based on all available information, and establish a system of differential equations for evaluating the statewise reserves.

2 The interest process

It is well-known that an amount, 1 say, invested at time zero in an economic environment where the force of interest δ is assumed constant, has a value at time t of

$$R_t = \exp(\delta\, t).$$

This is equivalent to solving the integral equation

$$R_t = 1 + \int_0^t R_s \delta ds,$$

and informally we can write

$$R_t = \prod_{0<s\le t} [1 + \delta ds].$$

The present value V_t of a unit payable at time t becomes

$$V_t = \exp(-\delta\, t).$$

However, the return of an investment (in e.g. shares or bonds) can be uncertain due to unforeseen events such as inflation and changes in market values of the asset. We shall represent this phenomenon in a stochastic process I_t governing the return process, and it will also be referred to as the interest process. It is assumed to be a cadlág (right-continuous paths possessing left-hand limits) process with paths of bounded variation over finite intervals, fulfilling $I_0 = 0$. It can then be decomposed into its continuous and discontinuous part, denoted I_t^c and I_t^d, respectively, as follows

$$I_t = I_t^c + \sum_{0<s\le t} I_s^d,$$

where $I_t^d = I_t - I_{t-}$ and $I_{t-} = \lim_{s\nearrow t} I_s$.

Assume now that I_t jumps at the times $T_1 < T_2 < \dots$ and let $Y_1, Y_2 \dots$ denote the corresponding size of the jumps. We obtain a sequence $(T_n, Y_n)_{n\ge1}$ called a marked point process, where T_n refer to the points and Y_n the marks. In the applications below, the mark will represent the random amount of price change of the asset.

Let $1(F)$ denote the indicator for a set $F \in \mathcal{F}$, and introduce the counting measures

$$N(t, A) = \sum_{n=1}^{\infty} 1(T_n \le t,\ Y_n \in A), \quad A \in \mathcal{B},$$

which counts the number of jumps over $[0, t]$ with marks in A, where $\mathcal{B}$ denotes the Borel σ-algebra on $\mathcal{R}$ the real line, and we define $N(t) = N(t, \mathcal{R})$. The interest process can then be written as

$$I_t = I_t^c + \int_{(0,t]} \int_{\mathcal{R}} y\, N(ds, dy),$$

meaning, in the applications below, that a random increase (decrease) in the price of the asset shall correspond to an upward (downward) jump in I_t. A random increase (decrease) in inflation shall then be represented as a decrease (increase) in the price. In the sequel, we will write $\int_a^b$ for $\int_{(a,b]}$.

The stochastic development of a unit invested at time zero is now assumed to be governed by the stochastic integral equation

$$\begin{aligned} X_t &= 1 + \int_0^t X_{s-} dI_s \\ &= 1 + \int_0^t X_{s-} \left[dI_s^c + \int_{\mathcal{R}} y\, N(ds, dy) \right]. \end{aligned} \tag{2.1}$$

The solution to (2.1) is given as (see e.g. Liptser and Shiryayev, 1989, p. 122)

$$\mathcal{E}_t(I) = \exp(I_t^c) \prod_{0<s\le t} \left[1 + \int_{\mathcal{R}} y\, N(ds, dy) \right], \tag{2.2}$$

which is called the Doléans exponential and (2.1) is called the Doléans equation. Whenever $\mathcal{E}_t(I) \neq 0$, the present value of a unit payable at time t is defined as $\mathcal{E}_t^{-1}(I)$, and by (2.2) it is seen to be given as

$$\mathcal{E}_t^{-1}(I) = \exp(-I_t^c) \prod_{0<s\le t} \left[1 - \int_{\mathcal{R}} \frac{y}{1+y} N(ds, dy) \right], \tag{2.3}$$

which is the solution to the Doléans equation

$$X_t = 1 - \int_0^t X_{s-} \left[dI_s^c + \int_{\mathcal{R}} \frac{y}{1+y} N(ds, dy) \right].$$

Using Newton's formula

$$(a+b)^n = \sum_{q=0}^{n} \begin{pmatrix} n \\ q \end{pmatrix} a^q b^{n-q}, \quad a, b \in \mathcal{R}, \quad n \geq 1,$$

we get by (2.2) that

$$\begin{aligned} \mathcal{E}_t^n(I) &= \exp(nI_t^c) \prod_{0<s\le t} \left[1 + \int_{\mathcal{R}} y\, N(ds, dy) \right]^n \\ &= \exp(nI_t^c) \prod_{0<s\le t} \left[1 + \sum_{q=1}^{n} \begin{pmatrix} n \\ q \end{pmatrix} \int_{\mathcal{R}} y^q N(ds, dy) \right], \end{aligned}$$

where we have used the property

$$\left(\int_{\mathcal{R}} y\, N(dt, dy) \right)^m = \int_{\mathcal{R}} y^m N(dt, dy), \quad m = 0, 1, 2 \ldots$$

Thus $\mathcal{E}_t^n(I)$ satisfies the Doléans equation

$$X_t = 1 + \int_0^t X_{s-} \left[n\, dI_s^c + \sum_{q=1}^{n} \begin{pmatrix} n \\ q \end{pmatrix} \int_{\mathcal{R}} y^q N(ds, dy) \right],$$

and similarly we obtain

$$\mathcal{E}_t^{-n}(I) = \exp(-nI_t^c) \prod_{0<s\le t} \left[1 + \sum_{q=1}^{n} \begin{pmatrix} n \\ q \end{pmatrix} \int_{\mathcal{R}} \left(\frac{-y}{1+y} \right)^q N(ds, dy) \right].$$

In the case where $Y_n \in (-1, \infty)$ a.s. (almost surely), we can also view the products above as exponentials. We illustrate for $n = 1$. Taking the logarithm we obtain

$$\begin{aligned} \log \left(\prod_{0<s\le t} \left[1 + \int_{\mathcal{R}} y\, N(ds, dy) \right] \right) &= \sum_{0<s\le t} \log \left(1 + \int_{\mathcal{R}} y\, N(ds, dy) \right) \\ &= \int_0^t \int_{\mathcal{R}} \log(1+y) N(ds, dy), \quad a.s. \end{aligned}$$

Thus

$$\prod_{0<s\le t}\left[1+\int_{\mathcal{R}} y\, N(ds,dy)\right]=\exp\left(\int_0^t\int_{\mathcal{R}}\log(1+y)N(ds,dy)\right),\quad a.s.,$$

and similarly for the product in (2.3), we can write

$$\begin{aligned}\prod_{0<s\le t}\left[1-\int_{\mathcal{R}}\frac{y}{1+y}N(ds,dy)\right] &= \exp\left(\int_0^t\int_{\mathcal{R}}\log(1-\frac{y}{1+y})N(ds,dy)\right)\\ &= \exp\left(-\int_0^t\int_{\mathcal{R}}\log(1+y)N(ds,dy)\right),\quad a.s.\end{aligned}$$

However, such relations seem not of particular importance for evaluating the mean values of $\mathcal{E}_t^{\,n}(I)$ and $\mathcal{E}_t^{\,-n}(I)$.

For further analysis we need the concept of an intensity process. Introduce the natural filtration

$$\mathcal{F}_t^N=\sigma(N(s,A),\ s\le t,\ A\in\mathcal{B}),$$

and assume that $N(t,A)$ admits a cadlag $\mathcal{F}_t^N$-intensity process $\lambda_t(A)$ informally given by

$$\lambda_t(A)dt=E[N(dt,A)\,|\,\mathcal{F}_{t-}^N]+o(dt),$$

where $o(h)/h\to 0$ as $h\to 0$ and $\mathcal{F}_{t-}^N=\vee_{s<t}\mathcal{F}_s^N$, is the information prior to time t.

It can be more convenient to write the intensity as

$$\lambda_t(A)=\lambda_t\int_A G_t(dy),$$

where G_t is a probability, $\int_{\mathcal{R}} G_t(dy)=1$, and λ_t is the intensity of $N(t)$. In particular, we get

$$E[N(t,A)]=E\left[\int_0^t\lambda_s(A)ds\right],$$

whenever $E[N(t,A)]<\infty$. More generally we know that (see Brémaud, 1981, p. 235)

$$M_t=\int_0^t\int_{\mathcal{R}}H(s,y)(N(ds,dy)-\lambda_s(dy)ds),$$

is a zero mean $\mathcal{F}_t^N$-martingale, where H is an $\mathcal{F}_t^N$-predictable process such that

$$E\left[\int_0^t\int_{\mathcal{R}}|H(s,y)|\lambda_s(dy)ds\right]<\infty.$$

In particular

$$E\left[\int_0^t\int_{\mathcal{R}}H(s,y)N(ds,dy)\right]=E\left[\int_0^t\int_{\mathcal{R}}H(s,y)\lambda_s(dy)ds\right].\tag{2.4}$$

In the sequel it should be sufficient to know that, in particular, any process with left-continuous or deterministic paths is predictable. For definitions of predictable σ-algebra and predictable processes we refer to Brémaud (1981, pp. 8, 9, 234, 235).

2.1 The independent increment case

In this subsection we will assume that the interest process has independent increments. This will be obtained by assuming that $(N(t,A))_{t\geq 0}$ has independent increments, which is equivalent to stating that $N(t,A)$ for each A is a Poisson process with the intensity being deterministic. Furthermore, we shall assume that I_t^c is of the form

$$I_t^c = \int_0^t \delta_s ds, \tag{2.5}$$

where δ_t is the force of interest, assumed to be a piecewise continuous function. As a consequence of the independent increment property of $N(t,A)$, G_t is given by

$$P(Y_n \in A \,|\, T_n = t) = \int_A G_t(dy), \quad \forall n \geq 1.$$

It is now possible using a direct approach to obtain expressions for the moments of the processes in (2.2) and (2.3). Define $r_t = E[\mathcal{E}_t(I)]$ and obtain by (2.1), (2.4) and (2.5) that

$$r_t = 1 + \int_0^t r_s \left[\delta_s + \int_{\mathcal{R}} y\,\lambda_s(dy)\right] ds,$$

whenever

$$\int_0^t \int_{\mathcal{R}} |y|\,\lambda_s(dy)ds < \infty,$$

which leads to the solution

$$r_t = \exp\left(\int_0^t \left[\delta_s + \int_{\mathcal{R}} y\,\lambda_s(dy)\right] ds\right). \tag{2.6}$$

Define $\tilde{r}_t = E[\mathcal{E}_t^{-1}(I)]$ and obtain as for (2.6) that

$$\tilde{r}_t = \exp\left(-\int_0^t \left[\delta_s + \int_{\mathcal{R}} \frac{y}{1+y}\lambda_s(dy)\right] ds\right), \tag{2.7}$$

whenever

$$\int_0^t \int_{\mathcal{R}} |\frac{y}{1+y}|\,\lambda_s(dy)ds < \infty.$$

Consequently, the expected value of $\mathcal{E}_T(I)$ is equivalent to finding the accumulated amount of 1 at time T, in an environment with the deterministic force of interest

$$\eta_t = \delta_t + \lambda_t E[Y_t], \quad E[Y_t] = \int_{\mathcal{R}} y\,G_t(dy).$$

Note that

$$r_t = \exp\left(\int_0^t \delta_s ds + E\left[\int_0^t \int_{\mathcal{R}} y\,N(ds,dy)\right]\right),$$

and

$$\tilde{r}_t = \exp\left(-\int_0^t \delta_s ds - E\left[\int_0^t \int_{\mathcal{R}} \frac{y}{1+y}\,N(ds,dy)\right]\right).$$

Define $r_t^n = E[\mathcal{E}_t^n(I)]$ and $\tilde{r}_t^n = E[\mathcal{E}_t^{-n}(I)]$, and use similar arguments for obtaining (2.6) and (2.7) to arrive at

$$r_t^n = \exp\left(\int_0^t \left[n\delta_s + \sum_{q=1}^n \binom{n}{q} \int_{\mathcal{R}} y^q \lambda_s(dy)\right] ds\right),$$

and

$$\tilde{r}_t^n = \exp\left(\int_0^t \left[-n\delta_s + \sum_{q=1}^n \binom{n}{q} \int_{\mathcal{R}} \left(\frac{-y}{1+y} \right)^q \lambda_s(dy) \right] ds \right),$$

whenever

$$\int_0^t \int_{\mathcal{R}} |y|^n \lambda_s(dy) ds < \infty, \quad \int_0^t \int_{\mathcal{R}} |\frac{y}{1+y}|^n \lambda_s(dy) ds < \infty.$$

A generalization of the independent increment assumption is to let I_t be a Markov process satisfying the equation

$$I_t = \int_0^t \delta_s(I_s) ds + \int_0^t \int_{\mathcal{R}} y \, N(ds, dy), \tag{2.8}$$

where $(t, x) \to \delta_t(x)$ is some realvalued function such that $t \to \delta_t(x)$ is interpreted as the force of interest allowed to depend on the present state of the interest process. Furthermore, the intensity process of $N(t, A)$ may depend on the past only via I_t. An example could be a version of the 'Ornstein-Uhlenbeck' process by considering the stochastic differential equation

$$dI_t = (\mu + \alpha(I_t - \mu))dt + \int_{\mathcal{R}} y \, N(dt, dy),$$

where μ, α are constants.

It seems then no longer feasible to make direct calculations for evaluating similar mean values as above, however it can be proved, by use of a similar martingale approach introduced in Møller (1993), that these can be identified from an integro-differential equation, which can be solved numerically. We will not pursue the process in (2.8) further, but proceed to the case below that leads to finite systems of differential equations, which is very convenient in a numerical implementation, and also the techniques used for deriving the equations can be based on the classical Kolmogorov's backward differential equations.

2.2 A Markovian environment

In this section we will introduce a heterogeneous environment for the economy given by different states, representing for instance different levels of inflation or different prices for a share or bond.

We consider the case of a finite number J of states, and assume that transitions between the states are governed by a Markov process $(\Theta_t)_{t \geq 0}$. The state space is denoted by $\mathcal{J} = \{1, 2, \ldots, J\}$.

We associate a marked point process, $(T_n, Z_n)_{n \geq 1}$ where T_n denote the jump times of Θ_t, and Z_n are the respective states entered, that is, $\Theta_{T_n} = Z_n$. Introduce the associated counting measures

$$N(t, j) = \sum_{n=1}^{\infty} 1(T_n \leq t, \ \Theta_{T_n} = j), \quad j \in \mathcal{J}.$$

We assume that there exist deterministic piecewise continuous functions $\lambda_{ij}(t)$, $i \neq j$, $i, j \in \mathcal{J}$, such that $N(t, j)$ admits the intensity process $\lambda_{\Theta_t j}(t) 1(\Theta_t \neq j)$. Also, we will assume that there are given deterministic piecewise continuous functions $\delta_i(t)$ and $\gamma_{ij}(t)$, such that the interest process is given by

$$I_t = \int_0^t \delta_{\Theta_s}(s) ds + \sum_{i \in \mathcal{J}} \int_0^t \gamma_{\Theta_{s-} i}(s) \, N(ds, i). \tag{2.9}$$

We write $\lambda_k(t) = \sum_{j \neq k} \lambda_{kj}(t)$, and below we shall write $\sum_i$ instead of $\sum_{i \in \mathcal{J}}$. The process in (2.9) tells, that the economy changes stochastically in accordance with a Markovian environment, and between the jumps, return on the investment is earned continuously and deterministically with a rate depending on the present state of Θ_t.

Throughout, we will assume that $\gamma_{ij}(t) \neq -1$, $i \neq j$, such that $\mathcal{E}_t^{-1}(I)$ exists for all t. Modifying (2.2) and (2.3) in accordance with (2.9), we get

$$\mathcal{E}_t(I) = \exp\left(\int_0^t \delta_{\Theta_s}(s)ds\right) \prod_{0<s\leq t}\left[1 + \sum_i \gamma_{\Theta_{s-}\,i}(s)N(ds,i)\right] \tag{2.10}$$

and

$$\mathcal{E}_t^{-1}(I) = \exp\left(-\int_0^t \delta_{\Theta_s}(s)ds\right) \prod_{0<s\leq t}\left[1 - \sum_i \frac{\gamma_{\Theta_{s-}\,i}(s)}{1+\gamma_{\Theta_{s-}\,i}(s)}N(ds,i)\right], \tag{2.11}$$

respectively. More generally for $n \geq 1$, we have

$$\mathcal{E}_t^n(I) = \exp\left(n\int_0^t \delta_{\Theta_s}(s)ds\right) \prod_{0<s\leq t}\left[1 + \sum_i \sum_{q=1}^n \begin{pmatrix} n \\ q \end{pmatrix} \gamma_{\Theta_{s-}\,i}(s)^q N(ds,i)\right]$$

and

$$\mathcal{E}_t^{-n}(I) = \exp\left(-n\int_0^t \delta_{\Theta_s}(s)ds\right) \prod_{0<s\leq t}\left[1 + \sum_i \sum_{q=1}^n \begin{pmatrix} n \\ q \end{pmatrix} \left(\frac{-\gamma_{\Theta_{s-}\,i}(s)}{1+\gamma_{\Theta_{s-}\,i}(s)}\right)^q N(ds,i)\right].$$

To evaluate the means of $\mathcal{E}_t^n(I)$ and $\mathcal{E}_t^{-n}(I)$, we can no longer make a direct (forward) approch, but instead we fix an interval $[0,T]$, $T<\infty$, and for identifying, $E[\mathcal{E}_T(I)\,|\,\Theta_0]$ say, we introduce the cadlág process $t \to \mathcal{E}_{(t,T]}(I)$ given by

$$\mathcal{E}_{(t,T]}(I) = \exp\left(\int_t^T \delta_{\Theta_s}(s)ds\right) \prod_{t<s\leq T}\left[1 + \sum_i \gamma_{\Theta_{s-}\,i}(s)N(ds,i)\right], \quad t \in [0,T],$$

where $\mathcal{E}_{(T,T]}(I) = 1$. Using integration by parts, it is readily checked that $\mathcal{E}_{(t,T]}(I)$ satisfies the backward Doléans equation

$$\mathcal{E}_{(t,T]}(I) = 1 + \int_t^T \mathcal{E}_{(s,T]}(I)dI_s.$$

Introduce the functions $Q_j(t)$ by

$$Q_j(t) = E[\mathcal{E}_{(t,T]}(I)\,|\,\Theta_t = j], \quad j \in \mathcal{J}.$$

In the following we abbreviate $E_{\Theta_t}[\quad] = E[\quad|\,\Theta_t]$. Taking conditional expectation $E[\quad|\,\mathcal{F}_t^N]$ and using the tower property $E[E[\quad|\,\mathcal{F}_t^N]\,|\,\mathcal{F}_s^N] = E[\quad|\,\mathcal{F}_s^N]$, $s \leq t$, together with the Markov property and (2.9), we obtain

$$\begin{aligned} Q_{\Theta_t}(t) &= 1 + E_{\Theta_t}\int_t^T Q_{\Theta_s}(s)dI_s \\ &= 1 + E_{\Theta_t}\int_t^T Q_{\Theta_s}(s)\delta_{\Theta_s}(s)ds + E_{\Theta_t}\int_t^T \sum_i Q_i(s)\gamma_{\Theta_{s-}\,i}(s)N(ds,i) \\ &= 1 + E_{\Theta_t}\int_t^T Q_{\Theta_s}(s)\delta_{\Theta_s}(s)ds + E_{\Theta_t}\int_t^T \sum_i Q_i(s)\gamma_{\Theta_s i}(s)\lambda_{\Theta_s i}(s)1(\Theta_s \neq i)ds, \end{aligned}$$

where the last equality sign is obtained using the martingale property of the process

$$M_t = \int_0^t \sum_i Q_i(s)\gamma_{\Theta_{s-}\,i}(s)[N(ds,i) - \lambda_{\Theta_s i}(s)1(\Theta_s \neq i)ds].$$

Introduce now the transition probabilities

$$P_{ij}(s,t) = P(\Theta_t = j \,|\, \Theta_s = i), \quad s \le t,$$

and arrive at

$$Q_k(t) = 1 + \sum_j \int_t^T P_{kj}(t,s) Q_j(s)\delta_j(s)ds + \sum_j \int_t^T P_{kj}(t,s) \sum_{i\neq j} Q_i(s)\gamma_{ji}(s)\lambda_{ji}(s)ds, \tag{2.12}$$

and finally together with Kolmogorov's backward differential equations

$$\frac{dP_{ij}(t,u)}{dt} = \lambda_i(t)\, P_{ij}(t,u) - \sum_{l\neq i} \lambda_{il}(t) P_{lj}(t,u), \quad i,j \in \mathcal{J}, \quad t \le u,$$

we can state:

Theorem 2.1 *Over the continuity points of $\lambda_{ij}(t)$, $\gamma_{ij}(t)$ and $\delta_i(t)$, the functions $Q_k(t)$ satisfy the system of differential equations*

$$\frac{dQ_k(t)}{dt} = [-\delta_k(t) + \lambda_k(t)\,]\, Q_k(t) - \sum_{i\neq k} \lambda_{ki}(t)[\,1 + \gamma_{ki}(t)\,]\, Q_i(t), \quad t \in (0,T), \quad k \in \mathcal{J}. \tag{2.13}$$

Proof: Follows by differentiation in (2.12).

The system in (2.13) is evaluated under the initial condition $Q_k(T) = 1$, $k \in \mathcal{J}$, and the mean of $\mathcal{E}_T(I)$ can then be found as

$$E[\mathcal{E}_T(I) \,|\, \Theta_0] = Q_{\Theta_0}(0).$$

Consequently, we can establish differential equations for evaluating the mean of $\mathcal{E}_T^n(I)$, $\forall n \ge 1$. Introduce the cadlág process

$$\begin{aligned} \mathcal{E}^n_{(t,T]}(I) \quad &= \quad \exp\left(n \int_t^T \delta_{\Theta_s}(s)ds \right) \\ &\qquad \times \prod_{t<s\le T} \left[1 + \sum_i \sum_{q=1}^n \begin{pmatrix} n \\ q \end{pmatrix} \gamma_{\Theta_{s-}\,i}(s)^q N(ds,i) \right], \end{aligned}$$

and again using integration by parts we now establish the backward Doléans equation

$$\mathcal{E}^n_{(t,T]}(I) = 1 + \int_t^T \mathcal{E}^n_{(s,T]}(I) dI_s^{(n)},$$

where

$$dI_t^{(n)} = n\delta_{\Theta_t}(t)dt + \sum_i \sum_{q=1}^n \begin{pmatrix} n \\ q \end{pmatrix} \gamma_{\Theta_{t-}\,i}(t)^q N(dt,i).$$

Introduce the functions $Q_j^{(n)}(t)$, defined as

$$Q_j^{(n)}(t) = E[\mathcal{E}^n_{(t,T]}(I) \,|\, \Theta_t = j], \quad j \in \mathcal{J},$$

where, of course, $Q_j^{(1)}(t) = Q_j(t)$. By virtue of the techniques leading to Theorem 2.1, we immediately get:

Theorem 2.2 *Over the continuity points of $\lambda_{ij}(t)$, $\gamma_{ij}(t)$ and $\delta_i(t)$, the functions $Q_k^{(n)}(t)$ satisfy the system of differential equations*

$$\begin{aligned}\frac{dQ_k^{(n)}(t)}{dt} &= \left[-n\delta_k(t)+\lambda_k(t)\right] Q_k^{(n)}(t) \\ &\quad -\sum_{i\neq k}\lambda_{ki}(t)\left(1+\sum_{q=1}^{n}\binom{n}{q}\gamma_{ki}(t)^q\right)Q_i^{(n)}(t), \quad t\in(0,T), \quad k\in\mathcal{J}.\end{aligned}$$

For evaluate the mean of $\mathcal{E}_T^{-n}(I)$, we introduce the process

$$\begin{aligned}\mathcal{E}_{(t,T]}^{-n}(I) &= \exp\left(-n\int_t^T \delta_{\Theta_s}(s)ds\right) \\ &\quad \times \prod_{t<s\leq T}\left[1+\sum_i\sum_{q=1}^{n}\binom{n}{q}\left(\frac{-\gamma_{\Theta_{s-}\,i}(s)}{1+\gamma_{\Theta_{s-}\,i}(s)}\right)^q N(ds,i)\right],\end{aligned}$$

and define the functions

$$\tilde{Q}_j^{(n)}(t) = E[\mathcal{E}_{(t,T]}^{-n}(I)\,|\,\Theta_t=j], \quad j\in\mathcal{J},$$

to stating:

Theorem 2.3 *Over the continuity points of $\lambda_{ij}(t)$, $\gamma_{ij}(t)$ and $\delta_i(t)$ the functions $\tilde{Q}_k^{(n)}(t)$ satisfy the system of differential equations*

$$\begin{aligned}\frac{d\tilde{Q}_k^{(n)}(t)}{dt} &= \left[\,n\delta_k(t)+\lambda_k(t)\right]\tilde{Q}_k^{(n)}(t) \\ &\quad -\sum_{i\neq k}\lambda_{ki}(t)\left(1+\sum_{q=1}^{n}\binom{n}{q}\left(\frac{-\gamma_{ki}(t)}{1+\gamma_{ki}(t)}\right)^q\right)\tilde{Q}_i^{(n)}(t), \quad t\in(0,T), \quad k\in\mathcal{J}.\end{aligned}$$

3 An extended model

As a further illustration we will extend the Markov model above by introducing piecewise continuous deterministic payment functions $b_i(t)$ and $b_{ij}(t)$, where $b_i(t)$ is the rate of payment (premium, pension) when the environment is in state i, and $b_{ij}(t)$ is a lump sum paid immediately upon a transition from state i to j. In this manner an insurance company has a way of adjusting its pension payment or premium in accordance with the current state of the economy. We treat pensions as negative payments, premium and lump sums as positive. For the sake of illustration and notational convenience we omit the possibility of a mortality rate.

Let $\mathcal{E}_t(I)$ be given by (2.10), which is the solution to the Doléans equation

$$dX_t = X_{t-}\left(\delta_{\Theta_t}(t)dt+\sum_i\gamma_{\Theta_{t-}\,i}(t)N(dt,i)\right).$$

Then, as discussed above, $\mathcal{E}_t^{-1}(I)$ is given by (2.11) and is the solution to

$$dX_t = -X_{t-}\left(\delta_{\Theta_t}(t)dt+\sum_i\frac{\gamma_{\Theta_{t-}\,i}(t)}{1+\gamma_{\Theta_{t-}\,i}(t)}N(dt,i)\right). \tag{3.1}$$

Let the stochastic surplus over the time interval $[0, t]$ be represented by the process R_t given by the stochastic differential equation

$$dR_t = R_{t-}\left(\delta_{\Theta_t}(t)dt + \sum_i \gamma_{\Theta_{t-}\,i}(t)N(dt,i)\right) + b_{\Theta_t}(t)dt - \sum_i b_{\Theta_{t-}\,i}(t)N(dt,i). \tag{3.2}$$

Under the condition $R_0 = 0$ it is readily checked that

$$R_t = \mathcal{E}_t(I)\int_0^t \mathcal{E}_s^{-1}(I)\left(b_{\Theta_s}(s)ds - \sum_i b_{\Theta_{s-}\,i}(s)N(ds,i)\right).$$

The stochastic (discounted) net expenses over the time interval $(t, T]$ is defined by

$$R_{(t,T]} = \mathcal{E}_t(I)\int_t^T \mathcal{E}_s^{-1}(I)\left(\sum_i b_{\Theta_{s-}\,i}(s)N(ds,i) - b_{\Theta_s}(s)ds\right), \quad t \in [0,T],$$

which is also obtained by solving (3.2) with initial condition zero at time T. An insurer is typically interested in the process

$$\begin{aligned} V_{\mathcal{F}_t^N}(t) &= E[R_{(t,T]}\,|\,\mathcal{F}_t^N] \\ &= \mathcal{E}_t(I)\,E\left[\int_t^T \mathcal{E}_s^{-1}(I)\left(\sum_i b_{\Theta_{s-}\,i}(s)N(ds,i) - b_{\Theta_s}(s)ds\right)\middle|\,\mathcal{F}_t^N\right], \quad t \in [0,T]. \end{aligned} \tag{3.3}$$

Defining $\mathcal{E}_{(t,s]}^{-1}(I) = \mathcal{E}_t(I)\mathcal{E}_s^{-1}(I)$, $t \le s$, we get

$$\mathcal{E}_{(t,s]}^{-1}(I) = \exp\left(-\int_t^s \delta_{\Theta_u}(u)du\right)\prod_{t<u\le s}\left[1 - \sum_i \frac{\gamma_{\Theta_{u-}\,i}(u)}{1+\gamma_{\Theta_{u-}\,i}(u)}N(du,i)\right], \quad t \le s,$$

so (3.3) reduces (the Markov property) to a function depending only on Θ_t, abbreviated $V_{\Theta_t}(t)$, where

$$V_i(t) = E\left[\int_t^T \mathcal{E}_{(t,s]}^{-1}(I)\left(\sum_i b_{\Theta_{s-}\,i}(s)N(ds,i) - b_{\Theta_s}(s)ds\right)\middle|\,\Theta_t = i\right], \quad t \in [0,T], \quad i \in \mathcal{J}$$

We can now use similar techniques as in Section 2 to establish a system of differential equations for evaluating $V_i(t)$. First, rewrite $R_{(t,T]}$ as

$$R_{(t,T]} = \mathcal{E}_t(I)\int_t^T \mathcal{E}_{s-}^{-1}(I)\left(\sum_i \frac{b_{\Theta_{s-}\,i}(s)}{1+\gamma_{\Theta_{s-}\,i}(s)}N(ds,i) - b_{\Theta_s}(s)ds\right), \quad t \in [0,T],$$

and using integration by parts, we observe that $t \to R_{(t,T]}$ satisfies

$$\begin{aligned} dR_{(t,T]} &= R_{(t,T]}\left(\delta_{\Theta_t}(t)dt + \sum_i \frac{\gamma_{\Theta_{t-}\,i}(t)}{1+\gamma_{\Theta_{t-}\,i}(t)}N(dt,i)\right) \\ &\quad - \sum_i \frac{b_{\Theta_{t-}\,i}(t)}{1+\gamma_{\Theta_{t-}\,i}(t)}N(dt,i) + b_{\Theta_t}(t)dt, \quad t \in [0,T], \end{aligned}$$

and therefore under the condition $R_{(T,T]} = 0$, we have

$$\begin{aligned} R_{(t,T]} &= -\int_t^T R_{(s,T]}\left(\sum_i \frac{\gamma_{\Theta_{s-}\,i}(s)}{1+\gamma_{\Theta_{s-}\,i}(s)}N(ds,i) + \delta_{\Theta_s}(s)ds\right) \\ &\quad + \int_t^T \left(\sum_i \frac{b_{\Theta_{s-}\,i}(s)}{1+\gamma_{\Theta_{s-}\,i}(s)}N(ds,i) - b_{\Theta_s}(s)ds\right), \quad t \in [0,T]. \end{aligned}$$

Using arguments similar to those for obtaining (2.12) we can write:

$$
\begin{aligned}
V_{\Theta_t}(t) &= -E_{\Theta_t} \int_t^T \left(\sum_i \frac{V_i(s)\gamma_{\Theta_{s-}\,i}(s)}{1+\gamma_{\Theta_{s-}\,i}(s)} N(ds,i) + V_{\Theta_s}(s)\delta_{\Theta_s}(s)ds \right) \\
&\quad + E_{\Theta_t} \int_t^T \left(\sum_i \frac{b_{\Theta_{s-}\,i}(s)}{1+\gamma_{\Theta_{s-}\,i}(s)} N(ds,i) - b_{\Theta_s}(s)ds \right) \\
&= -E_{\Theta_t} \int_t^T \left(\sum_i \frac{V_i(s)\gamma_{\Theta_s\,i}(s)}{1+\gamma_{\Theta_s\,i}(s)} \lambda_{\Theta_s i}(s) 1(\Theta_s \neq i) + V_{\Theta_s}(s)\delta_{\Theta_s}(s) \right) ds \\
&\quad + E_{\Theta_t} \int_t^T \left(\sum_i \frac{b_{\Theta_s\,i}(s)}{1+\gamma_{\Theta_s\,i}(s)} \lambda_{\Theta_s i}(s) 1(\Theta_s \neq i) - b_{\Theta_s}(s) \right) ds.
\end{aligned}
$$

Hence, using the transition probabilities, we get

$$
\begin{aligned}
V_k(t) &= - \sum_j \int_t^T P_{kj}(t,s) \left(\sum_{i\neq j} \frac{V_i(s)\gamma_{ji}(s)}{1+\gamma_{ji}(s)} \lambda_{ji}(s) + V_j(s)\delta_j(s) \right) ds \\
&\quad + \sum_j \int_t^T P_{kj}(t,s) \left(\sum_{i\neq j} \frac{b_{ji}(s)}{1+\gamma_{ji}(s)} \lambda_{ji}(s) - b_j(s) \right) ds, \quad t \in [0,T], \quad k \in \mathcal{J},
\end{aligned}
$$

and by differentiation we obtain:

Theorem 3.1 *Over the continuity points of $\lambda_{ij}(t)$, $\gamma_{ij}(t)$, $\delta_i(t)$, $b_i(t)$ and $b_{ij}(t)$ the functions $V_k(t)$ satisfy the system of differential equations*

$$
\frac{dV_k(t)}{dt} = [\,\delta_k(t) + \lambda_k(t)\,]\, V_k(t) + b_k(t) - \sum_{i\neq k} \lambda_{ki}(t) \left\{ \frac{b_{ki}(t) + V_i(t)}{1+\gamma_{ki}(t)} \right\}, \quad t \in (0,T), \quad k \in \mathcal{J}.
$$

References

[1] K. Aase, Contingent claims valuation when the security price is a combination of an Ito process and a random variable, *Stoc. Proc. Appl.* 28 (1988), 185-220.

[2] P. Brémaud, Point Processes and Queues, (Springer-Verlag, New-York, 1981)

[3] H. Bühlmann, Stochastic discounting, *Insurance: Math. and Economics* 11 (1992), 113-127.

[4] C.D. Daykin, T. Pentikäinen and M. Pesonen, Practical risk theory for actuaries, (Chapmann & Hall, 1994).

[5] H.M. Dietz, A stochastic interest model with an application to insurance, *Insurance: Math. and Economics* 11 (1992), 301-310.

[6] D. Dufresne, The distribution of perpetuity, with applications to risk theory and pension funding, *Scand. actuarial J.* (1990), 39-79.

[7] Liptser, R. and Shiryayev, A.N. (1989) *Theory of martingales*, Springer-Verlag, New-York.

[8] C.M. Møller, Asymptotic results for the risk process based on marked point processes, *Scand. Actuarial J.* 2 (1991), 169-184.

[9] C.M. Møller, A stochastic version of Thiele's differential equations, *Scand. Actuarial J.* 1 (1993), 1-16.

[10] C.M. Møller, Stochastic differential equations for ruin probabilities, *J. Appl. Prob.* 32 (1995), 74-89.

[11] J. Paulsen, Risk theory in a stochastic environment, *Stoc. Proc. Appl.* 46 (1993), 327-361.