Stochastic Differential Equations for Ruin Probabilities

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Abstract

The present paper proposes a general approach for finding differential equations to evaluate probabilities of ruin in finite and infinite time. Attention is given to real valued non-diffusion processes where a Markov structure is obtainable. Ruin is allowed to occur upon a jump or between the jumps. The key point is to define a process of conditional ruin probabilities and identify this process stopped at the time of ruin as a martingale. Using the theory of marked point processes together with the change of variable formula or (more elegantly) the martingale representation theorem for point processes, we obtain stochastic differential equations for evaluating the probability of ruin.

Numerical illustrations are given by solving a partial differential equation numerically to obtain the probability of ruin over a finite time horizon.

Keywords: *Point process, Martingale representation, Markov process, Markovian environment.*

1 Introduction

Over the years a lot of results on evaluation of ruin probabilities have appeared. One is interested in finding the probability of ruin in finite or infinite time as a function of the initial reserve. Several approaches are taken, hereunder upper bounds, diffusion approximations and differential equations. Differential equations for ruin probabilities have so far been studied mainly in the classical model, where the risk process is a compound Poisson process and the premium rate is constant. Under this model an integro-differential equation for the probability of ruin in infinite time can be derived by a renewal argument, see Grandell (1990, p. 4) or Feller (1971, p. 183). Extensions to models where the intensity and premium can fluctuate according to a Markovian environment, are discussed by Asmussen (1989) and Reinhard (1984). Asmussen and Petersen (1988) study a more refined model with reserve with premium depending on the reserve and, using the theory of storage processes, they obtain equations for the probability of ruin in infinite time. The risk reserve becomes a homogeneous Markov process, but need not have independent increments. Petersen (1989) gives some numerical examples for this model. Dassios and Embrechts (1989) use an approach based on formulas for the infinitesimal generator for PD (piecewise deterministic) Markov processes obtained by Davis (1984), and derive differential equations for evaluating upper bounds and explicit expressions of the probability of ruin. For finite time ruin, it is possible to establish a connection to their approach, see Paragraph 3.2.

In this paper we will focus on differential equations for ruin probabilities. The aim is to understand how one can establish differential equations for quite general models, and furthermore, how the generalizations obtained are related to the cases in the references listed. The key tool to obtain the equations is martingale theory. The complexity of the equations increases with the complexity of the model, and explicit formulas quickly become unattainable. Instead we put emphasis on numerical procedures and give some numerical results. The complexity of the models can be due to the stochastic nature of the portfolio (the intensities) which the company cannot control, or the policy of the company itself, such as reinsurance scheme and premium setting or (discrete) bonus payments.

We start by defining a stochastic process of conditional ruin probabilities where time varies over an interval $[0, T)$. More precisely, we will be interested in e.g. the probability of ruin over (t, T) given the reserve at time t.

These conditional probabilities lead to a process, and the key point is that this process, stopped at the time of ruin, becomes a martingale. Using this property, we can obtain (stochastic) differential equations for the process of conditional ruin probabilities, by use of the change of variable formula or the martingale representation theorem, where the latter is the most elegant and informative approach, see the comments following (**??**). In general it seems not possible to establish pure deterministic differential equations, see the comments following (3.7). On the other hand, when solving the equations numerically we think of them as deterministic.

We introduce here a concise and mathematical tractable presentation for studying ruin probabilities, which is suitable for quite general models. The approach by identifying the process of conditional ruin probabilities as a martingale seems to be new. Furthermore, this martingale property is derived using purely mathematical steps, and hence does not rely on any PD Markov structure or whatever. But, of course, the differential equations rely on the structure of the risk models.

We restrict to non-diffusion models, and the key tool is the theory of point processes with associated martingale theory. As will become apparent, this framework is appropriate for studying insurance models. The reader must be familiar with this theory and basic concepts as stopping times, predictable processes and predictable σ-algebra. Rigorous expositions of these topics can be found in e.g. Rogers and Williams (1987), Chung and Williams (1990), and Brémaud (1981).

In Section 2 we state some basic definitions and results from point process theory which can be found in Brémaud (1981). In Paragraph 3.1 we consider a non-homogeneous Markov risk reserve and derive a (stochastic) partial differential equation using the change of variable formula, which is equivalent to deriving the extended generator of the involved risk reserve process. The partial differential equation is transformed into a first order integro-differential equation, which can be solved numerically. The approach based on representations for martingales is briefly discussed as a more general and informative way of deriving the equations. In Paragraph 3.2, we discuss the results in relation to other approaches. In Paragraph 3.3, we give some examples of numerical evaluation of the partial differential equation in Section 2 to obtain the probability of ruin in finite time. In Section 4 we proceed to more complex models, where e.g. the premium or the claims intensity can fluctuate in such a way that the risk model is no longer Markov, but such that a Markovization is still possible. First we study a case where the intensity depends on the history only via the number of jumps and, second, we study the risk model in a Markovian environment.

2 Some elements of point process theory

It is assumed throughout that all random variables encountered are defined on some probability space $(\Omega, \mathcal{F}, P)$. A point process is a sequence $(T_n, Z_n)_{n\geq 1}$ of stochastic pairs, where $T_1, T_2, \ldots$ are non-negative and represent times of occurrence of some phenomena represented by the stochastic elements $Z_1, Z_2, \ldots$, called the marks, which are assumed to take values in some measurable space $\mathcal{Z}$ endowed with a σ-algebra $\mathcal{E}$. This model framework is ideal for studying insurance problems. In non-life insurance, the points typically represent times of occurrence of claims, and the marks represent the individual claim amounts. In such a case one is interested in the risk process

$$X_t = \sum_{n=1}^{N_t} Z_n,$$

which is the total amount of claims over a time interval $(0, t]$, where N_t is the number of claims over $(0, t]$. Another example is where the points represent times of notification of claims to the company, and the marks represent the pairs of the individual claim amount and the delay time from occurrence of the claim to notification. In life insurance the points could represent times of transition between states of a process governing the policy, and the state entered or the pair of states involved in the transition could represent the mark.

Let $\mathcal{R}$, $\mathcal{R}_+$ denote the real line and the non-negative half line, respectively, endowed with the usual Borel σ-algebras. Let $I(F)$ denote the indicator for a set F in $\mathcal{F}$.

We shall be interested in risk processes of the form

$$X_t^{(f)} = \sum_{n=1}^{N_t} f(T_n, Z_n),$$

where $f : \mathcal{R}_+ \times \mathcal{Z} \to \mathcal{R}$ is some Borel measurable mapping, and N_t represents the number of events in the time interval $(0, t]$. The process $X_t^{(f)}$ could e.g. represent the total amount of claims over $(0, t]$, where each claim amount is multiplied by a discount factor. Throughout the points will be referred to as the jump times.

In the sequel, $X_t^{(f)}$ will also be written

$$X_t^{(f)} = \int_{(0,t]} \int_{z \in \mathcal{Z}} f(s, z) dN_s(dz), \tag{2.1}$$

where $N_t(A)$ is the counting measure

$$N_t(A) = \sum_{i=1}^{\infty} I(T_i \le t,\, Z_i \in A), \tag{2.2}$$

which counts the number of jumps in the time interval $(0, t]$ with marks taking values in $A \in \mathcal{E}$. In particular $N_t = N_t(\mathcal{Z})$.

Throughout it is assumed that

(i) only a finite number of jumps can occur over finite intervals.

In the sequel, we will also write $\int_s^t$, $\int_{\mathcal{Z}}$ for $\int_{(s,t]}$, $\int_{z \in \mathcal{Z}}$, respectively.

The natural filtration is

$$\mathcal{F}_t^N = \sigma(N_s(A),\, s \le t,\, A \in \mathcal{E}).$$

Assume there is given a filtration $\mathcal{F}_t$ ($\mathcal{F}_t \supseteq \mathcal{F}_t^N$) such that $N_t(A)$ admits the $\mathcal{F}_t$-intensity $\lambda_t(A)$ assumed to be bounded over finite intervals, satisfying

$$\lambda_t(A)dt = E(dN_t(A) \,|\, \mathcal{F}_{t-}) + o(dt),$$

where $\mathcal{F}_{t-} = \vee_{s<t} \mathcal{F}_s$ is the information prior to time t. In the sequel we abbreviate $\lambda_t = \lambda_t(\mathcal{Z})$, which is the intensity for N_t. We can also write the intensity on the form

$$\lambda_t(A) = \lambda_t \int_A G_t(dz),$$

where G_t is a probability, $\int_{\mathcal{Z}} G_t(dz) = 1$, and $G_t(dz)$ is interpreted as the conditional probability given all prior information and that a jump occurred at time t, that the mark associated will belong to $(z, z + dz)$. An important result (e.g. Brémaud, 1981, p. 27) states that the process

$$M_t(A) = N_t(A) - \int_0^t \lambda_s(A) ds$$

for each $A \in \mathcal{E}$ is a zero mean $\mathcal{F}_t$-martingale whenever $E[N_t(A)] < \infty$, for $t > 0$. Also, one can integrate predictable processes w.r.t. the martingale in (**??**) and obtain martingales (Brémaud, 1981, pp. 27, 235). For definitions of the predictable σ-algebra and predictable processes we refer to Brémaud, 1981, pp. 8-9. In the sequel it should be sufficient to know that, in particular, any process with left continuous or deterministic paths is predictable.

3 The risk reserve as a Markov process

3.1 The martingale approach

We start with the following definition of the risk reserve at time t:

$$R_t = R_0 + \int_0^t b(s, R_s)ds - \int_0^t \int_{\mathcal{Z}} f(s,z)dN_s(dz). \tag{3.1}$$

The mapping $(t,r) \to b(t,r)$ from $\mathcal{R}_+ \times \mathcal{R}$ to $\mathcal{R}$ is assumed to be piecewise continuous in t and r, and could represent premium income or annuity payments to the insured. Since we allow $b(t,r)$ to admit negative values, ruin is not only caused by a jump. Ruin between jumps could e.g. happen if the company pays out pensions, or if it invests its reserve and gets a negative outcome of its investments. To prevent technical details from obscuring the presentation, we omit the possibility of payments at discrete times such as for instance lump sum payments governed by the size of R_t.

The risk reserve is a cadlag process (right continuous paths with left limits), and it is required throughout that R_t is $\mathcal{F}_t$-measurable. We assume that the intensity (**??**) is deterministic, which is equivalent to stating that $N_t(A)$ is a Poisson process for each A, and the interpretation of G_t in (**??**) becomes

$$\int_A G_t(dz) = P(Z_n \in A \,|\, T_n = t). \;\; \forall n.$$

Then, since $b(t, R_t)$ depends on the history only through R_t, the reserve R_t becomes an $\mathcal{F}_t$-Markov process (it need not have independent increments), that is $\sigma(R_s,\, s \geq t)$ and $\mathcal{F}_t$ are independent given R_t. Also, we could allow f to depend on R_{t-}, since this will not destroy the Markov property of R_t and the mathematical steps leading to Theorem 3.1 below. For notational convenience we omit this.

The time of ruin is defined as

$$\tau = \inf\{t > 0 \,|\, R_t < 0\},$$

which is an $\mathcal{F}_t$-stopping time, and thus fulfills $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. For a fixed $T \leq \infty$ we write

$$\begin{aligned} I(\tau < T) &= I(\tau \leq t) + I(t < \tau < T) \\ &= I(\tau \leq t) + I(\tau > t) I(\inf_{t \leq s < T} R_s < 0) \\ &= I(\tau < t) + I(\tau \geq t) I(\inf_{t \leq s < T} R_s < 0). \end{aligned} \tag{3.2}$$

Defining $M_t = P(\tau < T \,|\, \mathcal{F}_t) = E(I(\tau < T) \,|\, \mathcal{F}_t)$, taking conditional expectation w.r.t. $\mathcal{F}_t$ in (3.2), and using the Markov property, we get

$$\begin{aligned} M_t &= I(\tau \leq t) + P(t < \tau < T \,|\, \mathcal{F}_t) \\ &= I(\tau \leq t) + I(\tau > t) P(\inf_{t \leq s < T} R_s < 0 \,|\, \mathcal{F}_t) \\ &= I(\tau \leq t) + I(\tau > t) \Psi(t, R_t) \\ &= I(\tau < t) + I(\tau \geq t) \Psi(t, R_t), \end{aligned} \tag{3.3}$$

where the function $\Psi : \mathcal{R}_+ \times \mathcal{R} \to [0,1]$ is

$$\Psi(t,r) = P(\inf_{t \leq s < T} R_s < 0 \,|\, R_t = r),$$

the probability of ruin after time t with reserve r at time t. Also, we get

$$\Psi(t,r) = 1, \;\; r < 0.$$

If we assume that $\Psi(t,r)$ is continuous in t,r, we obtain by (3.3) that M_t is right-continuous with left-hand limits given by

$$M_{t-} = I(\tau < t) + I(\tau \geq t)\Psi(t, R_{t-}).$$

In the sequel, we chose the version of Ψ such that M_t is cadlag.

Inserting $t \wedge \tau$ in (3.3), we get

$$M_{t\wedge\tau} = \Psi(t\wedge\tau, R_{t\wedge\tau}),\ \ t \in [0,T),$$

which in particular gives that $\Psi(t\wedge\tau, R_{t\wedge\tau})$ is a (uniformly integrable) martingale, and using this property we will derive a differential equation for the non-ruin probability

$$\Phi(t, R_t) = 1 - \Psi(t, R_t).$$

As outlined in the introduction, we see that the martingale property in (**??**) is derived using only the Markov property, and not the particular functional structure of R_t.

The function $\Psi(t,r)$ is independent of t if e.g. R_t is a homogenuous Markov process and $T=\infty$, and becomes then identical to the probability of ruin in infinite time with initial reserve r. With a finite time horizon or e.g. time dependent intensities the dependence on t cannot in general be suppressed. In the numerical procedures for evaluating ruin probabilities, we primarily operate with $T<\infty$ since it is then possible to state the initial condition $\Phi(T,r)=1$ for all r, and then solve $\Phi(t,r)$ over $[0,T)$ for some values of r.

Another relation: Using that $M_{t\wedge\tau} = E(I(\tau < T \,|\, \mathcal{F}_{t\wedge\tau})$ (optional sampling), we obtain by taking conditional expectation on both sides in (**??**) w.r.t. the $\mathcal{F}_{t\wedge\tau}$-measurable stochastic variable $(t\wedge\tau, R_{t\wedge\tau})$, that

$$P(\tau < T \,|\, t\wedge\tau, R_{t\wedge\tau}) = \Psi(t\wedge\tau, R_{t\wedge\tau}).$$

We now introduce the technique that leads to the differential equations. We assume that $\Phi(t,r)$ has continuous partial derivatives for $t,r>0$, which are denoted $\frac{\partial\Phi}{\partial t}(t,r)$ and $\frac{\partial\Phi}{\partial r}(t,r)$, respectively. The change of variable formula yields for $0<t<T$:

$$\begin{aligned}
&\Phi(t,R_t) - \Phi(0,R_0)\\
&= \int_0^t \frac{\partial\Phi}{\partial t}(s,R_s)ds + \int_0^t \frac{\partial\Phi}{\partial r}(s,R_s)b(s,R_s)ds\\
&\quad + \sum_{s\leq t}[\Phi(s,R_s) - \Phi(s,R_{s-})]\\
&= \int_0^t \frac{\partial\Phi}{\partial t}(s,R_s)ds + \int_0^t \frac{\partial\Phi}{\partial r}(s,R_s)b(s,R_s)ds\\
&\quad + \int_0^t\int_{\mathcal{Z}}[\Phi(s,R_{s-} - f(s,z)) - \Phi(s,R_{s-})]dN_s(dz)\\
&= \int_0^t \frac{\partial\Phi}{\partial t}(s,R_s)ds + \int_0^t \frac{\partial\Phi}{\partial r}(s,R_s)b(s,R_s)ds\\
&\quad + \int_0^t\int_{\mathcal{Z}}[I(R_s\geq f(s,z))\Phi(s,R_s - f(s,z)) - \Phi(s,R_s)]\lambda_s(dz)ds + M_t^*,
\end{aligned} \tag{3.4}$$

where

$$M_t^* = \int_0^t\int_{\mathcal{Z}}[I(R_{s-}\geq f(s,z))\Phi(s,R_{s-} - f(s,z)) - \Phi(s,R_{s-})]dM_s(dz),$$

and $M_t(A)$ is given by (**??**). In the last equality sign in (3.4) we have used (**??**): $\Phi(t, R_t) = 0$ for $R_t < 0$, and we have replaced R_{t-} with R_t when integration is w.r.t. Lebesgue measure. Since R_{t-} is left-continuous and f is deterministic, the integrand in (**??**) is $\mathcal{F}_t$-predictable, hence M^*_t is a zero mean $\mathcal{F}_t$-martingale (Brémaud, 1981, p. 235). Furthermore, $M^*_{t\wedge\tau}$ is a $\mathcal{F}_t$-martingale, since it appears by multiplying the left-continuous process $I(\tau \geq s)$ to the integrand in (**??**). Using that $\Phi(t\wedge\tau, R_{t\wedge\tau})$ is a martingale, we then get that

$$\begin{aligned}
&\Phi(t\wedge\tau, R_{t\wedge\tau}) - \Phi(0, R_0) - M^*_{t\wedge\tau} \\
&= \int_0^{t\wedge\tau} \frac{\partial\Phi}{\partial t}(s, R_s)ds + \int_0^{t\wedge\tau} \frac{\partial\Phi}{\partial r}(s, R_s)b(s, R_s)ds - \int_0^{t\wedge\tau} \Phi(s, R_s)\lambda_s ds \\
&+ \int_0^{t\wedge\tau} \int_{\{z|\, R_s \geq f(s,z)\}} \Phi(s, R_s - f(s,z))\lambda_s(dz)ds
\end{aligned} \tag{3.5}$$

is a zero mean $\mathcal{F}_t$-martingale. Obviously (3.5) is continuous and of bounded variation, which implies that it is constant and therefore equal to zero, see Chung and Williams (1990, pp. 87-88) or Rogers and Williams (1987, p. 54). Thus

$$\begin{aligned}
&\int_0^{t\wedge\tau} \frac{\partial\Phi}{\partial t}(s, R_s)ds + \int_0^{t\wedge\tau} \frac{\partial\Phi}{\partial r}(s, R_s)b(s, R_s)ds \\
&= \int_0^{t\wedge\tau} \Phi(s, R_s)\lambda_s ds - \int_0^{t\wedge\tau} \int_{\{z\,|\, R_s \geq f(s,z)\}} \Phi(s, R_s - f(s,z))\lambda_s(dz)ds.
\end{aligned} \tag{3.6}$$

These arguments lead to the main result of this section:

Theorem 3.1 *The process $\Psi(t\wedge\tau, R_{t\wedge\tau})$ is a (uniformly integrable) martingale. Over the continuity points of $\lambda_t(A)$, f and R_t, $\Phi(t, R_t) = 1 - \Psi(t, R_t)$ satisfies, for $t \in (0, \tau)$, the (stochastic) partial integro-differential equation*

$$\begin{aligned}
&\frac{\partial\Phi}{\partial t}(t, R_t) + \frac{\partial\Phi}{\partial r}(t, R_t)b(t, R_t) \\
&\quad = \Phi(t, R_t)\lambda_t - \int_{\{z\,|\, R_t \geq f(t,z)\}} \Phi(t, R_t - f(t,z))\lambda_t(dz).
\end{aligned} \tag{3.7}$$

Equation (3.7) implies that $\Phi(t, R_t)$, for $t \in (0, \tau)$, satisfies $\tilde{\mathcal{A}}\Phi = 0$, where $\tilde{\mathcal{A}}$ is the extended generator of R_t. Remark that $\Phi(t, R_t)$ is not a martingale, and therefore we cannot prove (3.7) for all t.

When evaluating (3.7) numerically, we think of it as deterministic equation, and it is convenient to transform it into a first order integro-differential equation. Firstly, let U_t be a non-negative function satisfying the differential equation

$$\frac{dU_t}{dt} = b(t, U_t),$$

meaning that U_t plays the role of R_t between the jumps. Using integration by parts on $\Phi(t, U_t)$, we can then use (3.7) to establish the (deterministic) integro-differential equation

$$d\Phi(t, U_t) = \Phi(t, U_t)\lambda_t dt - \int_{\{z\,|\, U_t \geq f(t,z)\}} \Phi(t, U_t - f(t,z))\lambda_t(dz)dt.$$

Furthermore, introduce the function $\tilde{\Phi}_u(t) \equiv \Phi(t, u + \int_0^t b(s, U_s))$, $u \in \mathcal{R}$, where $U_t = u + \int_0^t b(s, U_s)ds$, is referred to as the characteristic curve or just the characteristic, see e.g. Smith (1985, pp. 175-181). Using (**??**) and (**??**), we can then obtain the differential equation

$$\frac{d\tilde{\Phi}_u}{dt}(t) = \tilde{\Phi}_u(t)\lambda_t - \int_{\{z\,|\, U_t \geq f(t,z)\}} \tilde{\Phi}_{u-f(t,z)}(t)\lambda_t(dz).$$

With the initial condition $\tilde{\Phi}_u(T) = 1$, for all $u \geq -\int_0^T b(s, U_s)ds$, and $T < \infty$, (**??**) must be solved numerically, see Paragraph 3.3 for an example.

When operating with point process histories we could alternatively derive (**??**) by finding the martingale representation of $\Phi(t \wedge \tau, R_{t\wedge\tau})$ using e.g. the technique described in Brémaud (1981, pp. 64-66). Combining (**??**) and the fact that the martingale in (3.5) is zero, we already know that the representation must have the form

$$\begin{aligned} &\Phi(t \wedge \tau, R_{t\wedge\tau}) - \Phi(0, R_0) \\ &= \int_0^{t\wedge\tau} \int_{\mathcal{Z}} [I(R_{s-} \geq f(s,z))\Phi(s, R_{s-} - f(s,z)) - \Phi(s, R_{s-}]dM_s(dz). \end{aligned} \tag{3.8}$$

The representation states that, for $t \in (0, \tau)$, $\Phi(t, R_t)$ between the jumps ($dN_t = 0$) develops in accordance with the differential equation (**??**), where R_t takes the role of U_t. Therefore, using the representation theorem, we can arrive directly at (**??**) and then (**??**), which is the essential equation in a numerical procedure. So it is not crucial to assume that Φ has continuous partial derivatives (governed by a vector field), which is required when operating with generators for PD Markov processes. For instance, we get by (**??**) that $t \to \Phi(t, U_t)$ is (absolutely) continuous and, over the continuity points of $\lambda_t(A)$ and f, is a differentiable function satisfying (**??**). Also, the representation theorem does not rely on any Markov structure, but holds for a general point process, see Brémaud (1981).

In this paper integration by parts is chosen for convenience only, and is applicable when considering processes which can be transformed into Markov processes. On the other hand, this approach is not that elegant and informative as the one based on the representation theorem.

In the following we focus on the homogeneous case. This is obtained by assuming that $b(t,r)$, $f(t,z)$, and $\lambda_t(A)$ are independent of t. The intensity (**??**) is then given by

$$\lambda_t(dz) = \lambda\, G(dz),$$

where $\lambda\,(> 0)$ is the intensity for the homogeneous Poisson process N_t, and G is the claims distribution for the i.i.d. (independent and identically distributed) random variables $Z_1, Z_2, \ldots$ We consider the reserve

$$R_t = R_0 + \int_0^t b(R_s)ds - \int_0^t \int_{\mathcal{R}} z\, dN_s(dz), \tag{3.9}$$

where in particular $\int_0^t \int_{\mathcal{R}} z\, dN_s(dz) = \sum_{i=1}^{N_t} Z_i$, is a compound Poisson process. Using the property of homogeneity, we get

$$\begin{aligned} \Psi(t,r) &= P(\inf_{t\leq s<T} R_s < 0 \,|\, R_t = r), \\ &= P(\inf_{0\leq s<T-t} R_s < 0 \,|\, R_0 = r) \\ &= P(\tau < T - t \,|\, R_0 = r). \end{aligned}$$

We can then as well consider $\Phi^* = 1 - \Psi^*$, with

$$\Psi^*(t,r) = P(\tau < t \,|\, R_0 = r),$$

satisfying $\Psi^*(0,r) = 0$, for all $r \geq 0$. Since $\Psi^*(t,r) = \Psi(T-t, r)$ on $(0,T]$, we can then by virtue of Theorem 3.1 state:

Corollary 3.2 *Suppose R_t is a homogeneous Markov process given by (3.9). Then over the continuity points of R_t, $\Phi^*(t,R_t) = 1 - \Psi^*(t,R_t)$ satisfies, for $t \in (0,\tau)$, the (stochastic) partial integro-differential equation*

$$\begin{aligned} &-\frac{\partial \Phi^*}{\partial t}(t,R_t) + \frac{\partial \Phi^*}{\partial r}(t,R_t)b(R_t) \\ &\quad = \Phi^*(t,R_t)\lambda - \lambda \int_{\{z\,|\,R_t \geq z\}} \Phi^*(t, R_t - z)G(dz). \end{aligned} \tag{3.10}$$

Having evaluated Φ^* (numerically) over $(0, T]$, $T < \infty$, we can evaluate quantities such as $E[\tau I(\tau < T) \,|\, R_0 = r]$ and $E[\tau \wedge T \,|\, R_0 = r]$, and higher moments. For instance

$$\begin{aligned} E[\tau \wedge T \,|\, R_0 = r] &= \int_0^T \Phi^*(s, r) ds \\ &= \int_0^T \Phi(s, r) ds, \end{aligned} \tag{3.11}$$

where the first equality sign follows by definition, and the second follows from $\Phi^*(t, r) = \Phi(T - t, r)$.

By defining $\Psi(r) = \lim_{t\uparrow\infty} \Psi^*(t, r)$, we can, by cancelling the differentiation w.r.t. t in (3.10), obtain an integro-differential equation for the non-ruin probability in infinite time (compare with the comments preceding (**??**)),

$$\Phi(r) = 1 - \Psi(r).$$

We can now state:

Corollary 3.3 *Suppose R_t is a homogeneous Markov process given by (3.9). Then $\Psi(R_{t\wedge\tau})$ is a (uniformly integrable) martingale. Over the continuity points of R_t, $\Phi(R_t) = 1 - \Psi(R_t)$ satisfies, for $t \in (0, \tau)$, the (stochastic) integro-differential equation*

$$\frac{d\Phi}{dr}(R_t) b(R_t) = \lambda \Phi(R_t) - \lambda \int_{\{z \,|\, R_t \geq z\}} \Phi(R_t - z) G(dz).$$

Asmussen and Petersen (1988) have established an integral equation for evaluating the probability of ruin in infinite time, when R_t is given by (3.9) with $b(r) > 0$, and $Z_i > 0$. Equation (**??**) does not appear in Asmussen and Petersen (1988) or Petersen (1989), but can easily be derived using their connection between the stationary density of the content dam process and the non-ruin probability. If, furthermore, $b(r)$ is independent of r, (**??**) leads to the well-known equation for the classical model, see e.g. Grandell (1990, p. 4).

There is one case where we can shift to a homogeneous Markov process when evaluating the probability of ruin in infinite time for a non-homogeneous Markov process:

Example 3.1. The classical model under discounting.
When considering infinite time ruin, (3.7) can be reduced to an ordinary differential equation in r under suitable stationarity conditions: The risk process $X_t = \sum_{n=1}^{N_t} Z_n$ is a compound Poisson process with underlying intensity given by (**??**). Consider the economic environment where payments are discounted in accordance with a constant force of interest $\delta \neq 0$, so that the annual discount factor is $v = e^{-\delta}$. Premiums are paid continuously at a constant rate $b > 0$. The present value by time 0 of the surplus is given by

$$R_t = R_0 + b \int_0^t v^s ds - \sum_{n=1}^{N_t} v^{T_n} Z_n,$$

which is a time non-homogenuous $\mathcal{F}_t^N$-Markov process, and is of the form (3.1) with $b(t, r) \equiv bv^t$, $f(t, z) \equiv v^t z$. Due to the stationarity property of X_t and δ and since $T = \infty$, it should then be obvious that

$$\Psi(t, r) = \Psi(0, r/v^t), \tag{3.12}$$

which means that the probability of getting ruined after time t with reserve r is equivalent to getting ruined at time zero with initial reserve r/v^t. Therefore

$$\frac{\partial \Phi}{\partial t}(t, r) = r\delta v^{-t} \frac{\partial \Phi}{\partial r}(0, r/v^t). \tag{3.13}$$

Putting $t = 0$ in (3.7) and (3.13) we get

$$(b + \delta r) \frac{\partial \Phi}{\partial r}(0, r) = \lambda \Phi(0, r) - \lambda \int_0^r \Phi(0, r - z) G(dz),$$

which solves $\Phi(0, r)$ for any $r \geq 0$, see also Delbaen and Haezendonck (1987, pp. 105-107). Consequently, the ruin probability in an economic environment as described above can also be viewed as a ruin probability for a risk reserve consisting of the reserve dependent premium $b(r) = b + \delta r$ and risk process X_t. Obviously, this means that the ruin probability is reduced in the presence of positive interest.

This result holds over any time horizon $[0, T)$ also for a time dependent force of interest and for a general X_t process, but then (3.12) is no longer valid. This can be seen as follows: Let $\delta(t)$ be the time dependent force of interest assumed to be piecewise continuous, and let $\tilde{R}_t$ be the risk reserve consisting of the reserve dependent premium $b(t, r) = b(t) + \delta(t)r$, such that $\tilde{R}_t$ fulfills the differential equation

$$d\tilde{R}_t = b(t)dt + \delta(t)\tilde{R}_t dt - dX_t,$$

which has the solution

$$\tilde{R}_t = e^{\int_0^t \delta(s)ds}\{\tilde{R}_0 + \int_0^t b(s)e^{-\int_0^s \delta(u)du}ds - \int_0^t e^{-\int_0^s \delta(u)du}dX_s\}.$$

The first time $\tilde{R}_t$ becomes negative over $[0, T)$ is equivalent to the first time the process in parentheses becomes negative, which is the risk reserve in the economic environment with interest rate $\delta(t)$ and premium rate $b(t)$. $\square$

3.2 The martingale approach and its relation to other results

Below we shall discuss the results in the light of the set-up in Dassios and Embrechts (1989). The discussion is only applicable for $T < \infty$. They focus on a process Y_t, consisting of two components (η_t, Q_t), where η_t is set to 1 if ruin has not occurred by time t, otherwise 0, and Q_t is equal to the risk reserve whenever it is positive and $\eta_t = 1$, otherwise ($\eta_t = 0$) Q_t is defined to be absorbed in 0. By finding h in the domain of the extended generator $\mathcal{A}$ for the Y_t process satisfying $\mathcal{A}h(t, y) = 0$, they obtain that $h(t, Y_t)$ becomes a martingale. Using an approach similar to that in Dassios and Embrechts (1989, p. 187), we conclude that $h(t, Y_t) = h(t, 1, Q_t)I(Q_t > 0)$ is a martingale, where $h(t, 1, r)$ satisfies and equation similar to (3.7) for all $t, r > 0$, and $h(t, 0, 0) = 0$. If, furthermore $T < \infty$ and h is chosen such as $h(T, 1, r) = 1$ for all $r > 0$, then by conditioning on $(\tau < T)$ and $(\tau \geq T)$, and using the martingale property of $h(t, Y_t)$, we can obtain

$$P(\tau \geq T \,|\, R_0 = r) = h(0, 1, r),\ \forall r > 0,$$

where we define $P(\tau \geq T \,|\, R_0 = 0) = \lim_{r \downarrow 0} h(0, 1, r)$ for $t \geq 0$. There seems not to be a unique interpretation of $h(t, 1, r)$ except, of course, at time 0 and T. However, by use of the martingale property of $h(t, Y_t)$, we obtain that

$$h(t, 1, R_t)I(\tau > t) = P(\tau \geq T \,|\, R_t)I(\tau > t),$$

and then using (**??**) we also obtain that

$$h(t, 1, R_t)I(\tau > t) = \Phi(t, R_t)I(\tau > t).$$

The functions $\Phi(t, r)$ and $P(\tau \geq T \,|\, R_t = r)$, $T < \infty$, are not necessarily identical, since $\Phi(t, r)$ does not satisfy (3.7) for all $t, r > 0$. We can only obtain a stochastic differential equation for Φ.

It is not convenient to operate with the process $\Phi(t, R_t)I(\tau > t)$ in the steps leading to (3.5). The presentation of the results becomes more understandable and informative by observing the identity in (**??**), which is the key point in this paper.

3.3 Numerical illustrations

In this paragraph we shall give some examples of numerical evaluation of (**??**). We discretize the integral term by use of a Simpson formula and solve the respective system of differential equations recursively over $[0, T)$ by use of the classical Runge-Kutta method.

We consider a risk reserve of the form

$$R_t = R_0 + c\,t - \sum_{i=1}^{N_t} Z_i,$$

where $R_0 \geq 0$ is the initial reserve, $c > 0$ is the constant premium rate, and we assume that the intensity is given by (**??**), where G is assumed to possess a continuous density g on $\mathcal{R}_+$. Then (**??**) reads for $t \in (0, T)$

$$\frac{d\tilde{\Phi}_u}{dt}(t) = \tilde{\Phi}_u(t)\lambda - \lambda \int_{\{z \mid u+c\,t \geq z\}} \tilde{\Phi}_{u-z}(t)\, g(z) dz.$$

For a fixed $U > 0$, we want to evaluate $\tilde{\Phi}_u(0)$ for some $u \in [0, U]$. Let $0 = t_0 < t_1 < \cdots < t_m = T$, be a partition of the interval $[0, T]$ with an equidistant division norm $h = t_i - t_{i-1}$. Starting with the initial condition $\tilde{\Phi}_u(T) = 1$, for all $u \geq -cT$, we evaluate recursively $\tilde{\Phi}_u(t_i)$ for $u \in [-c\,t_i, U]$, $i = m-1, m-2, \ldots, 0$, where the intervals $[-c\,t_i, U]$ also are divided into subintervals with division norm h.

For an example, we assume that G is an exponential distribution function with mean 1, implying that

$$g(x) = e^{-x}, \;\; x \geq 0.$$

*Table 1: Values of $\tilde{\Psi}_r(0) = 1 - \tilde{\Phi}_r(0)$ under (**??**), (**??**), (**??**) with parameters*

$U = 15,\ T = 1$

r	$\lambda = 10$, $c = 11$, $h = 10$	$\lambda = 10$, $c = 11$, $h = 20$	$\lambda = 20$, $c = 22$, $h = 20$
0	0.794830	0.790128	0.835602
1	0.620574	0.616550	0.693935
2	0.475711	0.472393	0.570218
3	0.358370	0.355709	0.463685
4	0.265552	0.263470	0.373199
5	0.193722	0.192128	0.297356
6	0.139244	0.138048	0.234598
7	0.098692	0.097811	0.183309
8	0.069027	0.068390	0.141891
9	0.047674	0.047220	0.108829
10	0.032534	0.032217	0.082729
11	0.021951	0.021733	0.062344
12	0.014651	0.014503	0.046587
13	0.009678	0.009580	0.034528
14	0.006330	0.006266	0.025386
15	0.004101	0.004060	0.018521

The figures in Table 1 were evaluated by running a simple Pascal program on a Personal Computer. The choices of the parameters correspond to a safety loading on the premium of

$$\rho = 1 - \frac{c}{\lambda E[Z_1]} = 0.1.$$

Unfortunately, when comparing columns 1 and 2, there seems to be an inaccuracy, especially for smaller values of r.

4 More complex models

In this section we will discuss some models of relevance to insurance where the risk reserve is not necessarily Markov, but where a Markovization is feasible. We will study cases where the intensity is allowed to be a stochastic process.

Firstly, we will extend the model assumptions from Section 3 by allowing the intensity to depend on the history only via N_{t-}. This is indicated by writing $\lambda_t(N_{t-}, A)$. An example could be the following: Assume N_t is a homogeneous Poisson process with intensity λ and independent of $Z_1, Z_2, \ldots$, which also are assumed mutually independent such that Z_n is assumed to be distributed according to G_n. The intensity would then become

$$\lambda_t(N_{t-}, dz) = \lambda\, G_{N_{t-}+1}(dz).$$

We will consider the more general version of (3.1):

$$R_t = R_0 + \int_0^t b(s, R_s, N_s)ds - \int_0^t \int_{\mathcal{Z}} f(s,z)dN_s(dz), \tag{4.1}$$

where for each n, $(t,r) \to b(t,r,n)$ takes the role of b in (3.1). We could also allow f to depend on N_{t-}, but for simplicity we avoid this. Then (R_t, N_t) becomes a Markov process, and the martingale in (**??**) reads

$$M_{t\wedge\tau} = \Psi(t \wedge \tau, R_{t\wedge\tau}, N_{t\wedge\tau}), \tag{4.2}$$

where $\Psi(t,r,n) = P(\inf_{t\le s<T} R_s < 0 \,|\, R_t = r, N_t = n)$. The function $\Phi(t,r,n) = 1 - \Psi(t,r,n)$ will only change in t and r between the jumps and is assumed to be governed by the continuous partial derivatives, $\frac{\partial\Phi}{\partial t}(t,r,n)$ and

$\frac{\partial \Phi}{\partial r}(t, r, n)$, respectively. Repeating the technique used in Section 3, we obtain the following partial (stochastic) integro-differential equation for $\Phi(t, R_t, N_t)$, $t \in (0, \tau)$:

$$\begin{aligned} & \frac{\partial \Phi}{\partial t}(t, R_t, N_t) + \frac{\partial \Phi}{\partial r}(t, R_t, N_t) b(t, R_t, N_t) \\ = \quad & \Phi(t, R_t, N_t)\lambda_t(N_t) - \int_{\{z \mid R_t \geq f(t,z)\}} \Phi(t, R_t - f(t, z), N_t + 1)\lambda_t(N_t, dz). \end{aligned} \tag{4.3}$$

Using arguments similar to those following (**??**), (4.3) implies that for a non-negative function U_t satisfying the differential equation

$$\frac{dU_t}{dt} = b(t, U_t, n), \; n \geq 0,$$

$\Phi(t, U_t, n)$ satisfies the (deterministic) differential equation

$$d\Phi(t, U_t, n) = \Phi(t, U_t, n)\lambda_t(n)dt - \int_{\{z \mid U_t \geq f(t,z)\}} \Phi(t, U_t - f(t, z), n + 1)\lambda_t(n, dz)dt,$$

which leads to the following system of integro-differential equations in n:

$$\frac{d\tilde{\Phi}_u}{dt}(t, n) = \tilde{\Phi}_u(t, n)\lambda_t(n) - \int_{\{z \mid U_t \geq f(t,z)\}} \tilde{\Phi}_{u - f(t,z)}(t, n + 1)\lambda_t(n, dz),$$

where $\tilde{\Phi}_u(t, n) \equiv \Phi(t, u + \int_0^t b(s, U_s, n)ds, n)$, and $U_t = u + \int_0^t b(s, U_s, n)ds$. The initial condition becomes $\tilde{\Phi}_u(T, n) = 1$, for all $u \geq -\int_0^T b(s, U_s, n)ds$ and $n \geq 0$, and system (**??**) must finally be solved numerically. This problem shall not be pursued here.

Finally, we will study the probability of ruin in a Markovian environment. The set-up here is more general than that treated in Reinhard (1984) and Asmussen (1989) since the processes involved can be of non-homogeneous Markov type and, further, the premium is allowed to depend on the environment and reserve.

Assume there is given an (observable) Markov jump process $(\Theta_t)_{t \geq 0}$, which for simplicity is assumed to have finite state space $\mathcal{J} = \{1, \ldots, J\}$ satisfying $\Theta_0 = 1$. The intensity is assumed to fluctuate according to Θ_t, that is, it is assumed to be a function of Θ_t. To study this model in the framework of a point process, we let the marks represent either the pair (Y_i, θ_i), where Y_i is some random variable, assumed for simplicity to be non-negative and typically represents a claim amount, occurred at time T_i such that $\Theta_{T_i} = \theta_i$, or represent the pair of states (θ_{i-1}, θ_i), caused by a transition of Θ_t at time T_i. To compare the results here with those in Reinhard (1984), we will assume that these two kinds of event cannot coincide. We write $N(t, A)$ instead of $N_t(A)$ and decompose it into the two associated counting processes

$$N_i(t, B) = \sum_{k \geq 1} I(T_k \leq t, \; Y_k \in B, \; \Theta_{T_k} = i), \; B \in \mathcal{B}_+, \tag{4.4}$$

$$N_{ij}(t) = \sum_{k \geq 1} I(T_k \leq t, \; \Theta_{T_k} = j, \; \Theta_{T_k -} = i), \; i \neq j, \tag{4.5}$$

where $\mathcal{B}_+$ is the Borel σ-algebra on $\mathcal{R}_+$. The natural filtration $\mathcal{F}_t^N$ can now be considered as generated by (4.4) and (4.5). As mentioned above, we assume that (4.4) and (4.5) cannot have common jumps.

The intensities of the counting processes (4.4), (4.5) are denoted $\lambda_i(t, B)$ and $\lambda_{ij}(t)$, respectively, and are assumed to depend on the history only via Θ_t.

Consider the risk reserve

$$R_t = R_0 + \int_0^t b_{\Theta_s}(s, R_s)ds - \sum_{i \in \mathcal{J}} \int_0^t \int_{\mathcal{R}_+} f_i(s, y)dN_i(s, dy),$$

where $b_i(t,r)$ takes the role of $b(t,r)$ in (3.1). Then (R_t, Θ_t) becomes a non-homogeneous Markov process, which implies that the martingale in (**??**) reads

$$M_{t\wedge\tau} = \Psi_{\Theta_{t\wedge\tau}}(t\wedge\tau, R_{t\wedge\tau}),$$

where $\Psi_i(t,r) = P(\inf_{t\le s<T} R_s < 0 \,|\, R_t = r,\, \Theta_t = i)$.

Using the change of variable formula for $\Phi_{\Theta_t}(t,R_t) = 1 - \Psi_{\Theta_t}(t,R_t)$, we obtain

$$\begin{aligned}
&\Phi_{\Theta_t}(t,R_t) - \Phi_{\Theta_0}(0,R_0)\\
&= \int_0^t \frac{\partial\Phi_{\Theta_s}}{\partial t}(s,R_s)ds + \int_0^t \frac{\partial\Phi_{\Theta_s}}{\partial r}(s,R_s)b_{\Theta_s}(s,R_s)ds\\
&\quad + \sum_i \int_0^t \int_{\mathcal{R}_+} [I(R_{s-} \ge f_i(s,y))\Phi_i(s,R_{s-} - f_i(s,y)) - \Phi_i(s,R_{s-})]dN_i(s,dy)\\
&\quad + \sum_{i\ne j} \int_0^t [\Phi_i(s,R_s) - \Phi_j(s,R_s)]dN_{ij}(s),
\end{aligned} \tag{4.6}$$

where $\sum_{i\ne j} = \sum_i \sum_{j\ne i}$. It is used in (4.6) that the counting processes do not have common jumps, and in particular we have replaced R_{t-} with R_t in the last term. Put $\bar\lambda_i(t) = \sum_{j\ne i}\lambda_{ij}(t)$ and let $\lambda_i(t) = \lambda_i(t,\mathcal{R}_+)$. Repeating the arguments leading to Theorem 3.1, we can, for $t \in (0,\tau)$, obtain the following system of (stochastic) partial differential equations:

$$\begin{aligned}
&\frac{\partial\Phi_i}{\partial t}(t,R_t) + \frac{\partial\Phi_i}{\partial r}(t,R_t)b_i(t,R_t)\\
&= \Phi_i(t,R_t)\lambda_i(t) - \int_{\{y\,|\,R_t\ge f_i(t,y)\}} \Phi_i(t,R_t - f_i(t,y))\lambda_i(t,dy)\\
&\quad + \Phi_i(t,R_t)\bar\lambda_i(t) - \sum_{j\ne i}\Phi_j(t,R_t)\lambda_{ij}(t)\\
&= \Phi_i(t,R_t)(\lambda_i(t) + \bar\lambda_i(t)) - \int_{\{y\,|\,R_t\ge f_i(t,y)\}} \Phi_i(t,R_t - f_i(t,y))\lambda_i(t,dy)\\
&\quad - \sum_{j\ne i}\Phi_j(t,R_t)\lambda_{ij}(t).
\end{aligned} \tag{4.7}$$

In particular one sees that (4.7) reduces to an equation similar to (3.7) if Θ_t can admit only a single value.

An example of $\lambda_i(t,B)$ could be

$$\lambda_i(t,dy) = \lambda_i(t)F_i(dy),$$

where $\lambda_i(t)$ can be interpreted as a claims intensity for $N_i(t) \equiv N_i(t,\mathcal{R}_+)$, and F_i as a claim amount distribution depending on the state of Θ_t, where $N_i(t)$ and the claim amounts are conditionally independent given $\Theta_t = i$. Reinhard (1984) studied this case with $f_i(t,y) \equiv y$, and $\lambda_i(t)$, $\lambda_{ij}(t)$ independent of t and $b_i(t,r)$ independent of t and r.

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