Point Processes and Martingales in Risk Theory

$$\sum_{n=1}^{N_t} \mathbf{f}(\mathbf{T_n}, \mathbf{Z_n}) = \int_0^t \int_{\mathcal{Z}} \mathbf{f}(\mathbf{s}, \mathbf{z}) \mathbf{dN_s}(\mathbf{dz})$$

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PhD thesis at the University of Copenhagen

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Submitted April 1994

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April 1994

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Contents

Chapter 1

Introduction

In this thesis we will introduce parts of the modern theory of stochastic calculus and study its applications in risk theory.

In life and non-life insurance one is typically interested in evaluating certain functionals of processes describing the insurance business. A simple example could be the problem of evaluating the mean or higher order moments of jump (risk) processes or perhaps more gernerally, the mean of transforms of jump processes.

Other functionals arise as conditional expectations, for instance the expected present value of future (net) expenses evaluated on basis of some available information. This is relevant in connection with evaluation of premiums and of reserves or the liabilities of the insurance business. In life insurance, Thiele's differential equation is a celebrated tool for evaluating premium and reserves. The policy is typically assumed to be governed by a Markov process of finite state space. It is then possible to obtain a tractable expression for the state reserves in terms of integrals depending on the transitions probabilities and payments. Then using the nice structure of Kolmogorov's differential equations, one can establish a set of differential equations for the state reserves, see Hoem (1969). However if we allow for more complicated models, for instance by assuming that the transition intensities or the payments depend on the duration in the visiting state, it seems not tractable to go via transition probabilities. Since we are primarily interested in the reserve process itself, it would be more convenient to arrive directly at the differential equations for the state reserves. This is indeed possible, and we will introduce the idea and techniques to achieve this. This opens the possibility of introducing more complex insurance products taking further variables into account in the evaluation basis, such as for instance stochastic interest or the accumulated (stochastic) surplus.

Another important quantity in risk theory, useful in solvency assessments of a company, is the probability of ruin, which is defined mathematically as the first time the risk business becomes negative. The problem is to evaluate this probability over some period of time (finite or infinite) as a function of the initial reserve. A way of evaluting this probability is to establish its associated partial integro-differential equation, and then solve it numerically. In classical risk theory where the involved processes have stationary and independent increments, one can heuristically derive the differential equations. But for more complicated processes we must be more precise and also understand how to obtain the generalizations. This problem will also be treated.

In Chapter 2 we outline the basic mathematics used throughout. The building stones are delivered by the theory of marked point processes, hereunder counting process theory. This is an ideal tool for studying insurance models, since most phenomena of interest in insurance occur at random times with a stochastic event associated.

To obtain useful results, we will further apply the highly developed theory of martingales and stochastic integration. This elegant theory, makes it possible to give a remarkably general and informative presentation of important issues in risk theory.

A common line in the thesis is to identify a martingale in the system involving the functional of interest. By finding an integral representation of it, we can understand the properties of the functional, hereunder identify a (stochastic) differential equation for evaluating the functional.

In Chapter 3, we study the application of this idea to obtain general versions of Thiele's differential equation. The chapter is a revised version of the results in [23].

Chapter 4 treats the change of variable formula, which can be helpful for evaluating means of transforms of processes or to arrive at differential equations for different functionals, but the techniques are not essential for the analysis in this thesis. We give some examples of applications, hereunder derivation of an integro-differential equation for compound distribution functions.

Chapter 5 treats the problem of establishing differential equations for functionals measuring the probability of ruin.

Chapter 6 treats examples of establishing diffusion approximations for a risk business, hereunder for jump processes. These results could be combined with those in Chapter 5 to find differential equations for evaluating the probability of ruin when the risk business is approximated by a diffusion process. Nevertheless, it is beyond the scope of the thesis to treat this.

Chapter 2

Some elements of point process theory

All random variables encountered in the sequel are assumed to be defined on some probability space $(\Omega, \mathcal{F}, P)$.

Basic in the studies are the concept of marked point processes and the associated martingale theory. A point process is a sequence $(T_n, Z_n)_{n \geq 1}$ of stochastic pairs, where $T_1, T_2, \ldots$ are non-negative and represent times of occurrence of some phenomena represented by the stochastic elements $Z_1, Z_2, \ldots$, called the marks, which are assumed to take values in some measurable space $\mathcal{Z}$ endowed with a σ-algebra $\mathcal{E}$. This model framework is ideal for studying insurance problems. In non-life insurance, the points typically represent times of occurrence of claims, and the marks represent the individual claim amounts. In this case one is interested in the risk process

$$X_t = \sum_{n=1}^{N_t} Z_n,$$

which is the total amount of claims over a time interval $(0, t]$, where N_t is the number of claims over $(0, t]$. Another example is where the marks represent the delay time from occurrence of the claim to notification, possibly combined with the individual claim amount. In life insurance the points could represent times of transition between states of a process governing the policy, and the state entered or the pair of states involved in the transition could represent the mark.

Let $\mathcal{R}^n$ denote the n-dimensional euclidian space, where for $n = 1$ we abbreviate $\mathcal{R} = \mathcal{R}^1$. The non-negative half line is denoted $\mathcal{R}_+$. The spaces are endowed with their usual Borel σ-algebras, where $\mathcal{B}$ and $\mathcal{B}_+$ denote the Borel σ-algebra on $\mathcal{R}$ and $\mathcal{R}_+$, respectively. Let $I(F)$ denote the indicator of a set F in $\mathcal{F}$.

We shall mainly operate with jump (risk) processes of the form

$$X_t^{(f)} = \sum_{n=1}^{N_t} f(T_n, Z_n),$$

where $f : \mathcal{R}_+ \times \mathcal{Z} \to \mathcal{R}$ is some Borel measurable mapping, and N_t represents the number of events in the time interval $(0, t]$. The process $X_t^{(f)}$ could e.g. represent the total amount of claims over $(0, t]$, where each claim amount is multiplied by a discount factor. Throughout the points will be referred to as the jump times.

A key point in the sequel is to rewrite $X_t^{(f)}$ according to the stochastic integral

$$X_t^{(f)} = \int_{(0,t]} \int_{z \in \mathcal{Z}} f(s, z) dN_s(dz), \tag{0.1}$$

where $N_t(A)$ is the counting measure

$$N_t(A) = \sum_{i=1}^{\infty} I(T_i \leq t,\, Z_i \in A), \tag{0.2}$$

which counts the number of jumps in the time interval $(0, t]$ with marks taking values in $A \in \mathcal{E}$. In particular $N_t = N_t(\mathcal{Z})$. The counting processes lead to the natural filtration

$$\mathcal{F}_t^N = \sigma(N_s(A),\, s \leq t,\, A \in \mathcal{E}).$$

Another important issue is the existence of an intensity process: Assume there is given a filtration $\mathcal{F}_t$ ($\mathcal{F}_t \supseteq \mathcal{F}_t^N$) such that $N_t(A)$ admits an $\mathcal{F}_t$-intensity $\lambda_t(A)$ assumed to be bounded over finite intervals, satisfying

$$\lambda_t(A)dt = E(dN_t(A) \,|\, \mathcal{F}_{t-}) + o(dt), \tag{0.3}$$

where $\mathcal{F}_{t-} = \vee_{s<t} \mathcal{F}_s$ is the information prior to time t. We abbreviate $\lambda_t = \lambda_t(\mathcal{Z})$, which is the intensity of N_t. We can also write the intensity on the form

$$\lambda_t(A) = \lambda_t \int_A G_t(dz), \tag{0.4}$$

where G_t is a probability, $\int_{\mathcal{Z}} G_t(dz) = 1$, and is interpreted as the conditional probability given all information prior to time t and that a jump occurred at time t, that the associated mark will belong to $(z, z + dz)$. An important result (e.g. Brémaud, 1981, p. 27) states that the process

$$M_t(A) = N_t(A) - \int_0^t \lambda_s(A) ds \tag{0.5}$$

for each $A \in \mathcal{E}$ is a zero mean $\mathcal{F}_t$-martingale whenever $E[N_t(A)] < \infty$, $t > 0$. Also, one can integrate predictable processes w.r.t. the martingale in (0.5) and obtain martingales (Brémaud, 1981, pp. 27, 235). For definition of the predictable σ-algebra and predictable processes we refer to Brémaud, 1981, pp. 8-9, 235). In the sequel it should be sufficient to know that, in particular, any process with left continuous or deterministic paths is predictable. Consequently, we obtain that

$$\begin{aligned} M_t &= X_t^{(f)} - \int_0^t \int_{z \in \mathcal{Z}} f(s,z)\lambda_s(dz)ds \\ &= \int_{(0,t]} \int_{z \in \mathcal{Z}} f(s,z)(dN_s(dz) - \lambda_s(dz)ds), \end{aligned} \tag{0.6}$$

becomes a zero mean $\mathcal{F}_t$-martingale whenever

$$E\left[\int_0^t \int_{z \in \mathcal{Z}} |f(s,z)|\lambda_s(dz)ds\right] < \infty.$$

The process

$$C_t = \int_0^t \int_{z \in \mathcal{Z}} f(s,z)\lambda_s(dz)ds,$$

is called the compensator of $X_t^{(f)}$.

A constructive theorem used throughout is the following (Brémaud 1981, p. 239):

Theorem 0.1 *Any $\mathcal{F}_t^N$-martingale M_t admits a representation of the form*

$$M_t = M_0 + \int_{(0,t]} \int_{z \in \mathcal{Z}} H(s,z)(dN_s(dz) - \lambda_s(dz)ds), \tag{0.7}$$

where H is some $\mathcal{F}_t^N$-predictable process such that

$$\int_{(0,t]} \int_{z \in \mathcal{Z}} |H(s,z)|\lambda_s(dz)ds < \infty, \quad a.s.$$

In the sequel we will also write $\int_s^t \int_{\mathcal{Z}}$ instead of $\int_{(s,t]} \int_{z \in \mathcal{Z}}$.

Chapter 3

Expected values in life insurance (Thiele's differential equation)

In this chapter we present the idea of establishing quite general differential equation for evaluating premiums and reserves in life insurance. The idea is illustrated by some examples of interest in insurance.

Section 1 treats the classical Markov model, where the policy is assumed to be governed by a Markov process of finite state space. Also a formula for solving the equivalence premium and the state reserves numerically is discussed.

Section 2 treats a more general model where the transition intensities and the payments may depend on the duration in the visiting state. A numerical example with qualifying period for disabled lives for a three state Markov model is considered.

To illustrate further, we finally treat a case where the intensities and payments also could depend on the number of jumps occurred.

Section 1 is overall a special case of Section 2, but it may be easier to understand the technique used when starting with a less complex model with simpler notation. We will then in Section 2 allow, by reference to Section 1, skipping calculations that are similar. In Section 2 it seems not possible in general to establish formulas for evaluating the state reserves and equivalence premium.

3.1 The Markov model

The development of an insurance policy issued at time 0, say, is described by a time inhomogeneous Markov process X_t with finite state space $\mathcal{J} = \{1, \ldots, J\}$. The process is assumed to be cadlag (right continuous with left limits) and starts out in state 1; $X_0 = 1$.

Let $T_1 < T_2 < \ldots$ denote the jump times of the process, and let $Z_1, Z_2, \ldots$ be the states entered at the corresponding jump times and correspond to the marks. Define $T_0 = 0$ and $Z_0 = 1$.

We will operate with the multidimensional counting process $(N_{ij}(t))_{(i,j)\in\mathcal{J}}$ defined by

$$N_{ij}(t) = \sum_{n=1}^{\infty} I(T_n \leq t,\ (Z_{n-1}, Z_n) = (i,j)),$$

which counts the number of transitions from i to j in the time interval $(0,t]$. Introduce also $Y_i(t) = I(X_t = i)$. The intensity process (0.3) in Chapter 2 is determined by

$$P(dN_{ij}(t) = 1 \,|\, \mathcal{F}_{t-}^N) = \lambda_{ij}(t)Y_i(t)dt, \ \ i \neq j,$$

where $t \to \lambda_{ij}(t)$ are deterministic functions and the natural filtration is generated by $(N_{ij})_{(i,j)\in\mathcal{J}}$.

Consider now a classical life insurance policy with the following terms:

(a) Premiums and annuity benefits to the insured are paid continuously with rate $b_i(t)$, whenever the policy is in state i. The mappings $t \to b_i(t)$ from $\mathcal{R}_+$ to $\mathcal{R}$ are assumed to be piecewise continuous and non-stochastic. Premiums are negative and annuity benefits are positive.

(b) A non-stochastic lump sum of $b_{ij}(t)$ is paid to the insured immediately upon a transition from i to j at time t. The mappings $t \to b_{ij}(t)$ from $\mathcal{R}_+$ into $\mathcal{R}$ are assumed Borel measurable.

For simplicity, we assume that the force of interest δ is constant. Put $v = e^{-\delta}$, which is the annual discount factor. In the following we write $\sum_i$ and $\sum_{i\neq j}$ instead of $\sum_{i\in\mathcal{J}}$ and $\sum_{i\in\mathcal{J}}\sum_{j\neq i}$, respectively.

Fix a time $T < \infty$, and define the present value at time 0 of the stochastic surplus V_0 by

$$\begin{aligned} V_0 &= \sum_i \int_0^T v^s b_i(s) Y_i(s) ds + \sum_{i\neq j} \int_0^T v^s b_{ij}(s) dN_{ij}(s) \\ &= V_{(0,t]} + V_{(t,T]}, \ \ t \leq T, \end{aligned} \tag{1.1}$$

where

$$V_{(0,t]} = \sum_i \int_0^t v^s b_i(s) Y_i(s) ds + \sum_{i\neq j} \int_0^t v^s b_{ij}(s) dN_{ij}(s),$$

$$V_{(t,T]} = \sum_{i\neq j} \int_t^T v^s b_{ij}(s) dN_{ij}(s) + \sum_i \int_t^T v^s b_i(s) Y_i(s) ds,$$

with the assumption that $E|V_0| < \infty$.

The prospective reserve process $V_{\mathcal{F}_t}(t)$ is defined as

$$V_{\mathcal{F}_t}(t) = E(\frac{1}{v^t} V_{(t,T]} \,|\, \mathcal{F}_t). \tag{1.2}$$

Due to the Markov property, the conditional expectation in (1.2) depends only on X_t, hence the abbreviation $V_{X_t}(t)$, where

$$V_i(t) = E(\frac{1}{v^t} V_{(t,T]} \,|\, X_t = i).$$

Put $M_t = E(V_0 \,|\, \mathcal{F}_t^N)$. Since $V_{(0,t]}$ is $\mathcal{F}_t^N$-measurable, we get by (1.1) and (1.2) that

$$M_t = V_{(0,t]} + v^t V_{X_t}(t). \tag{1.3}$$

Under the assumption that the state reserves $t \to V_i(t)$, $i \in \mathcal{J}$, are right-continuous functions with left-hand limits (see Theorem 1.2 below) we can then obtain by (1.3) that M_t is a cadlag martingale over $[0,T]$ with left-hand limits given by

$$M_{t-} = V_{(0,t)} + v^t V_{X_{t-}}(t-).$$

To derive a stochastic differential equation for $V_{X_t}(t)$ we could use partial integration on $v^t V_{X_t}(t)$ in (1.3). The martingale M_t must finally be identified to obtain a useful differential equation.

Since v^t is continuous and of bounded variation, integration by parts gives (see e.g. Brémaud, 1981, p. 336)

$$\begin{aligned} d(v^t V_{X_t}(t)) &= d(v^t) V_{X_t}(t) + v^t dV_{X_t}(t) \\ &= -\delta dt\, v^t V_{X_t}(t) + v^t dV_{X_t}(t). \end{aligned} \tag{1.4}$$

Using (1.4), (1.3) implies

$$\begin{aligned} dM_t &= dV_{(0,t]} - \delta dt\, v^t V_{X_t}(t) + v^t dV_{X_t}(t) \\ &= \sum_i v^t b_i(t) Y_i(t) dt + \sum_{i \neq j} v^t b_{ij}(t) dN_{ij}(t) \\ &\quad -\delta dt\, v^t V_{X_t}(t) + v^t dV_{X_t}(t). \end{aligned} \tag{1.5}$$

Multiply by v^{-t} on both sides in (1.5) to obtain

$$dV_{X_t}(t) = \delta dt V_{X_t}(t) - \sum_i b_i(t)Y_i(t)dt - \sum_{i\neq j} b_{ij}(t)dN_{ij}(t) + v^{-t}dM_t. \quad (1.6)$$

To obtain a useful expression in (1.6), we need the following representation theorem:

Theorem 1.1 *Define the martingales* $M_{ij}(t)$ *by*

$$dM_{ij}(t) = dN_{ij}(t) - \lambda_{ij}(t)Y_i(t)dt,$$

and introduce

$$R_{ij}(t) = b_{ij}(t) + V_j(t) - V_i(t).$$

The martingale $M_t = E(V_0 \,|\, \mathcal{F}_t^N)$ *has the representation*

$$M_t = E[V_0] + \int_0^t \sum_{i\neq j} v^s R_{ij}(s)dM_{ij}(s),$$

whenever

$$\int_0^t \sum_{i\neq j} v^s |R_{ij}(s)| \lambda_{ij}(s)Y_i(s)ds < \infty, \quad a.s.$$

Proof: The technique of the proof is based on Brémaud (1981, pp. 64-68). Relation (1.3) becomes

$$M_t = \sum_i \int_0^t v^s b_i(s)Y_i(s)ds + \sum_{i\neq j} \int_0^t v^s b_{ij}(s)dN_{ij}(s) + v^t V_{X_t}(t).$$

Using $X_{T_n} = Z_n$ we also have

$$\begin{aligned}
M_{T_{n+1}} &= \sum_i \int_0^{T_{n+1}} v^s b_i(s)Y_i(s)ds + \sum_{i\neq j} \int_0^{T_{n+1}} v^s b_{ij}(s)dN_{ij}(s) \\
&\quad + v^{T_{n+1}} V_{X_{T_{n+1}}}(T_{n+1}) \\
&= \sum_{k=1}^{n+1} \int_{T_{k-1}}^{T_k} v^s b_{Z_{k-1}}(s)ds + \sum_{k=1}^{n} v^{T_k} b_{Z_{k-1}Z_k}(T_k) \\
&\quad + v^{T_{n+1}} b_{Z_n Z_{n+1}}(T_{n+1}) + v^{T_{n+1}} V_{Z_{n+1}}(T_{n+1}). \quad (1.7)
\end{aligned}$$

To obtain the representation, we should find $\mathcal{F}_{T_n}^N \otimes \mathcal{B}_+$-measurable mappings $(\omega, t) \to f^{(n)}(\omega, t, j)$, from $\Omega \times \mathcal{R}_+$ into $\mathcal{R}$ such that

$M_{T_{n+1}} = f^{(n)}(\omega, T_{n+1} - T_n, Z_{n+1})$.

An obvious candidate is

$$
\begin{aligned}
f^{(n)}(\omega, t, j) &= \sum_{k=1}^{n} \int_{T_{k-1}}^{T_k} v^s b_{Z_{k-1}}(s) ds + \int_{T_n}^{T_n+t} v^s b_{Z_n}(s) ds \\
&\quad + \sum_{k=1}^{n} v^{T_k} b_{Z_{k-1}Z_k}(T_k) + v^{T_n+t} b_{Z_n j}(T_n + t) \\
&\quad + v^{T_n+t} V_j(T_n + t). \qquad (1.8)
\end{aligned}
$$

For $T_n \leq t < T_{n+1}$, M_t is equal to

$$
\begin{aligned}
h^{(n)}(t) &= \sum_i \int_0^t v^s b_i(s) Y_i(s) ds + \sum_{i \neq j} \int_0^t v^s b_{ij}(s) dN_{ij}(s) + v^t V_{Z_n}(t) \\
&= \sum_{k=1}^{n} \int_{T_{k-1}}^{T_k} v^s b_{Z_{k-1}}(s) ds + \int_{T_n}^{t} v^s b_{Z_n}(s) ds \\
&\quad + \sum_{k=1}^{n} v^{T_k} b_{Z_{k-1}Z_k}(T_k) + v^t V_{Z_n}(t). \qquad (1.9)
\end{aligned}
$$

For $T_n \leq t < T_{n+1}$ (between the jump times), (Brémaud, 1981, p. 66), we get

$$
\begin{aligned}
M_t - M_{T_n} &= -\int_{T_n}^{t} \sum_{j \neq Z_n} [f^{(n)}(s - T_n, j) - h^{(n)}(s)] \lambda_{Z_n j}(s) ds \\
&= -\int_{T_n}^{t} \sum_{j \neq Z_n} v^s [b_{Z_n j}(s) + V_j(s) - V_{Z_n}(s)] \lambda_{Z_n j}(s) ds. \qquad (1.10)
\end{aligned}
$$

By letting $t \nearrow T_{n+1}$ in (1.9) and then subtracting the limit from (1.7), we get at the jump times

$$
\begin{aligned}
&M_{T_{n+1}} - M_{T_{n+1}-} \\
&= v^{T_{n+1}} [b_{Z_n Z_{n+1}}(T_{n+1}) + V_{Z_{n+1}}(T_{n+1}) - V_{Z_n}(T_{n+1}-)]. \qquad (1.11)
\end{aligned}
$$

For $t \nearrow T_{n+1}$ in (1.10) we obtain

$$\begin{aligned} & M_{T_{n+1}-} - M_{T_n} \\ & \quad = \; -\int_{T_n}^{T_{n+1}} \sum_{j \neq Z_n} v^s [b_{Z_n j}(s) + V_j(s) - V_{Z_n}(s)] \lambda_{Z_n j}(s) ds. \end{aligned} \tag{1.12}$$

Adding (1.11) and (1.12) gives

$$\begin{aligned} M_{T_{n+1}} - M_{T_n} &= \int_{T_n}^{T_{n+1}} \sum_{j \neq Z_n} v^s [b_{Z_n j}(s) + V_j(s) - V_{Z_n}(s-)] dM_{Z_n j}(s) \\ &= \int_{T_n}^{T_{n+1}} \sum_{j \neq Z_n} v^s R_{Z_n j}(s) dM_{Z_n j}(s), \end{aligned} \tag{1.13}$$

where (1.13) follows by the continuity of the state reserves. Combining (1.10) and (1.13), we get for any $t > 0$

$$\begin{aligned} M_t - E[V_0] &= M_t - M_0 \\ &= \sum_{n=0}^{\infty} \int_0^t \sum_{j \neq Z_n} v^s R_{Z_n j}(s) I(T_n < s \leq T_{n+1}) dM_{Z_n j}(s) \\ &= \int_0^t \sum_{i \neq j} v^s R_{ij}(s) dM_{ij}(s), \end{aligned} \tag{1.14}$$

which ends the proof. $\square$

The martingale representation in (1.14) has also been derived by Ramlau-Hansen (1988) by application of Thiele's differential equation to a policy with terms similar to (a) and (b). In this paper, the purpose of the representation is quite the opposite, to obtain stochastic versions of Thiele's differential equation.

Using the representation of M_t we can now state:

Theorem 1.2 *Over the continuity points of $b_i(t)$, $b_{ij}(t)$ and $\lambda_{ij}(t)$, the functions $t \to V_i(t)$ are continuously differentiable, and satisfy the system of differential equations:*

$$\frac{dV_i}{dt}(t) = \delta\, V_i(t) - b_i(t) - \sum_{j \neq i} [b_{ij}(t) + V_j(t) - V_i(t)] \lambda_{ij}(t). \tag{1.15}$$

Proof: Using (1.6) and (1.14) we obtain (written in differential form)

$$\begin{aligned} dV_{X_t}(t) &= \delta dt\, V_{X_t}(t) - \sum_i b_i(t)Y_i(t)dt - \sum_{i\neq j} b_{ij}(t)dN_{ij}(t) \\ &\qquad + \sum_{i\neq j} R_{ij}(t)dM_{ij}(t). \end{aligned}$$

Then over the points where $X_t = X_{t-} = i$, we can write

$$dV_i(t) = \delta dt\, V_i(t) - b_i(t)dt - \sum_{j\neq i}[b_{ij}(t) + V_j(t) - V_i(t)]\lambda_{ij}(t)dt,$$

because $dN_{ij}(t) = 0$, $i \neq j$, and the process is in state i, $Y_i(t) = 1$. In this manner we can identify the state reserves by solving the system (1.15). $\square$

Equation (1.15) leads to the well known deterministic variation of Thiele's differential equation for the Markov case, see Hoem (1969).

In the following a formal solution to (1.15) will be discussed and a formula for numerical evaluation of the equivalence premium and the state reserves is derived. This approach is chosen for the numerical evaluation in Example 2.1 below. The following results can basically be seen in Barnett and Cameron (1985) or Davis (1977).

Assume, that $\lambda_{ij}(t)$, $b_i(t)$ and $b_{ij}(t)$ are all continuous functions on the interior of an interval I of $\mathcal{R}_+$.

Define $W_i(t) = v^t\, V_i(t)$, $i \in \mathcal{J}$, and obtain from (1.15)

$$\begin{aligned} \frac{dW_i}{dt}(t) &= \lambda_i(t)W_i(t) - \sum_{j\neq i}\lambda_{ij}(t)W_j(t) \\ &\qquad -v^t[b_i(t) + \sum_{j\neq i}\lambda_{ij}(t)b_{ij}(t)],\ t \in I, \end{aligned} \tag{1.16}$$

where $\lambda_i(t) = \sum_{j\neq i}\lambda_{ij}(t),\ i \in \mathcal{J}$.

Define the column vectors

$$\mathbf{W}(t) = (W_1(t), \ldots, W_n(t))',$$

$$\mathbf{H}(t) = (H_1(t), \ldots, H_n(t))',$$

where $H_i(t) = -v^t[b_i(t) + \sum_{j\neq i}\lambda_{ij}(t)b_{ij}(t)]$.

The system (1.16) can then be written

$$\frac{d\mathbf{W}}{dt}(t) = \mathbf{\Lambda}(t)\mathbf{W}(t) + \mathbf{H}(t),\ t \in I, \tag{1.17}$$

where $-\mathbf{\Lambda}(t) = (\lambda_{ij}(t))_{(i,j)\in\mathcal{J}}$ is the intensity matrix, with the understanding that $\lambda_{ii}(t) = -\sum_{j\neq i}\lambda_{ij}(t)$.

In the following it is assumed that (1.15) can be written on the form (1.17) such that $\mathbf{\Lambda}(t)$ has elements that are functions of the intensities and such that $\mathbf{H}(t)$ is independent of the state reserves, see e.g. Examples 1.1 and 2.1 below.

Normally, a system like (1.17) is solved by first considering the homogeneous system obtained by putting $\mathbf{H}(t) \equiv 0$. The unique solution $\hat{\mathbf{W}}(t)$ to the homogeneous system with initial condition $\hat{\mathbf{W}}(t_0) = \mathbf{x}_0$ for some fixed $t_0 \in I$, is of the form

$$\hat{\mathbf{W}}(t) = \hat{\mathbf{\Phi}}(t_0, t)\mathbf{x}_0, \quad t \in I,$$

where $\hat{\mathbf{\Phi}}(t_0, t)$ is a matrix called the fundamental matrix or the basic solution chosen such that $\hat{\mathbf{\Phi}}(t_0, t_0)$ equals the unit matrix $\mathbf{1}$. The fundamental matrix can also be identified as the limit of the sequence of the matrix functions

$$\begin{aligned}\hat{\mathbf{\Phi}}_k(t_0, t) \quad = \quad & \mathbf{1} + \int_{t_0}^{t} \mathbf{\Lambda}(u_1)du_1 + \int_{t_0}^{t}\int_{t_0}^{u_1} \mathbf{\Lambda}(u_1)\mathbf{\Lambda}(u_2)du_2du_1 \\ & + \cdots + \int_{t_0}^{t}\cdots\int_{t_0}^{u_{k-1}} \mathbf{\Lambda}(u_1)\ldots\mathbf{\Lambda}(u_k)du_k\ldots du_1,\end{aligned}$$

where $u_0 = t$, and the limit is w.r.t. to the matrix norm $\|A\| = \max_{1\leq i\leq p}\sum_{i=1}^{p}|a_{ij}|$, defined for an $p \times p$ matrix $A = (a_{ij})$. Thus

$$\hat{\mathbf{\Phi}}(t_0, t) = \mathbf{1} + \sum_{k=1}^{\infty}\int_{t_0}^{t}\cdots\int_{t_0}^{u_{k-1}} \mathbf{\Lambda}(u_1)\ldots\mathbf{\Lambda}(u_k)du_k\ldots du_1. \tag{1.18}$$

By (1.18) we in particular obtain

$$\frac{d\hat{\mathbf{\Phi}}}{dt}(s, t) = \mathbf{\Lambda}(t)\hat{\mathbf{\Phi}}(s, t), \;\; s, t \in I, \tag{1.19}$$

$$\frac{d\hat{\mathbf{\Phi}}}{ds}(s, t) = -\hat{\mathbf{\Phi}}(s, t)\mathbf{\Lambda}(s), \;\; s, t \in I. \tag{1.20}$$

The following Theorem is needed, (Davis, 1977, p. 105, or Barnett and Cameron, 1985, pp. 76-77).

Theorem 1.3 *The fundamental matrix $\hat{\mathbf{\Phi}}(s, t)$ with $\hat{\mathbf{\Phi}}(t, t) = \mathbf{1}$, $\forall t \in I$, is regular $\forall\, s, t \in I$, and its inverse fulfills the system of differential equation*

$$\frac{d\mathbf{\Phi}^*}{dt}(s, t) = -\mathbf{\Phi}^*(s, t)\mathbf{\Lambda}(t), \tag{1.21}$$

and $\mathbf{\Phi}^*$ *satisfies*

$$\mathbf{\Phi}^*(s,t) = \mathbf{\Phi}^*(s,u)\mathbf{\Phi}^*(u,t),\ \forall\, s,t,u \in I. \tag{1.22}$$

By (1.20) and (1.21) we immediately get the relation

$$\hat{\mathbf{\Phi}}^{-1}(s,t) = \hat{\mathbf{\Phi}}(t,s)\,,\forall\, s,t \in I. \tag{1.23}$$

If $-\mathbf{\Lambda}$ is the intensity matrix, then (1.21) is the Kolmogorov forward differential equations, which gives for $s \leq t$ that

$$\hat{\mathbf{\Phi}}^{-1}(s,t) = \mathbf{P}(s,t), \tag{1.24}$$

where $\mathbf{P}$ is the matrix of the transition probabilities.

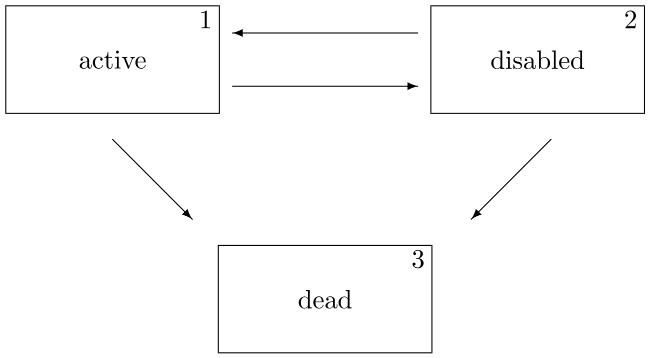
By e.g. multiplying with $\mathbf{\Phi}^{-1}(w,t)$ on both sides in (1.17) and first using (1.21), (1.23) and finally (1.22), we get that the solution to (1.17) with initial condition $\mathbf{W}(w) = v^w\,\mathbf{a}$, $w \leq \infty$, is given by

$$\begin{aligned}
\mathbf{W}(t) &= v^w\,\hat{\mathbf{\Phi}}(w,t)\mathbf{a} - \hat{\mathbf{\Phi}}(w,t)\int_t^w \hat{\mathbf{\Phi}}^{-1}(w,\tau)\mathbf{H}(\tau)d\tau \\
&= v^w\,\hat{\mathbf{\Phi}}^{-1}(t,w)\mathbf{a} - \hat{\mathbf{\Phi}}^{-1}(t,w)\int_t^w \hat{\mathbf{\Phi}}(\tau,w)\mathbf{H}(\tau)d\tau \\
&= v^w\,\hat{\mathbf{\Phi}}^{-1}(t,w)\mathbf{a} - \int_t^w \hat{\mathbf{\Phi}}^{-1}(t,\tau)\mathbf{H}(\tau)d\tau. \qquad (1.25)
\end{aligned}$$

In the case where $-\mathbf{\Lambda}$ is the intensity matrix we get by (1.24) that $\hat{\mathbf{\Phi}}^{-1}(t,\tau)$ in (1.25) equals $\mathbf{P}(t,\tau)$, and (1.25) could also in this case be obtained by a direct prospective argument for the state reserves.

Thus, the problem of evaluating the state reserves amounts to evaluating the matrix $\hat{\mathbf{\Phi}}^{-1}$, which normally must be done numerically, and then evaluating (1.25) numerically. Since $\hat{\mathbf{\Phi}}^{-1}$ has the multiplicative property (1.22), we can e.g. use product integral method to evaluate $\hat{\mathbf{\Phi}}^{-1}$, see [21] and Example 2.1. Alternatively, we can simply use classical methods for solving the homogeneous system (1.21). The equivalence premium is found from the equation $V_1(0) = 0$.

Example 1.1. The disability model.



The policy is described by the following terms: The insurance period is n years and premium is paid continuously with constant rate P for a period of at most m years, $m < n$, as long as the insured is in state 1. An annuity is paid continuously to the insured with constant rate b as long as the insured is in state 2. A benefit amount S, the sum insured, is paid immediately upon death within time n. Furthermore, the reserve $V_1(t)$ is paid to the insured if a transition from state 1 occurs at time t. Thus, $b_{12}(t) = V_1(t)$, $b_{13}(t) = S + V_1(t)$, $b_{23}(t) = S$ and $b_{21}(t) = 0$.

Since $V_3 \equiv 0$, it remains to find the equivalence premium and the state reserves, $V_i(t),\ i = 1, 2$, with initial condition $V_i(n) = 0$.

Under the assumptions above, (1.16) becomes

$$\frac{dW_1}{dt}(t) = -\lambda_{12}(t)W_2(t) + v^t[P - S\,\lambda_{13}(t)],$$

$$\frac{dW_2}{dt}(t) = \lambda_2(t)W_2(t) - \lambda_{21}(t)W_1(t) - v^t[S\,\lambda_{23}(t) + b],$$

with the interpretation $P = 0$ for $t \geq m$. This leads to the expression in (1.17) with

$$\mathbf{\Lambda}(t) = \begin{pmatrix} 0 & -\lambda_{12}(t) \\ -\lambda_{21}(t) & \lambda_2(t) \end{pmatrix},$$

and

$$\mathbf{H}(t) = -v^t(S\,\lambda_{13}(t) - P, S\,\lambda_{23}(t) + b)'.$$

With $\hat{\mathbf{\Phi}}^{-1} = (\psi_{ij})$ and $\mathbf{a} = \mathbf{0}$ formula (1.25) gives

$$\begin{aligned} V_1(t) &= \int_t^n v^{\tau-t}[\psi_{11}(t,\tau)S\,\lambda_{13}(\tau) + \psi_{12}(t,\tau)(S\,\lambda_{23}(\tau)+b)]d\tau \\ &\quad -P\int_t^m v^{\tau-t}\psi_{11}(t,\tau)d\tau, \\ V_2(t) &= \int_t^n v^{\tau-t}[\psi_{21}(t,\tau)S\,\lambda_{13}(\tau) + \psi_{22}(t,\tau)(S\,\lambda_{23}(\tau)+b)]d\tau \\ &\quad -P\int_t^m v^{\tau-t}\psi_{21}(t,\tau)d\tau. \end{aligned}$$

The equivalence premium is determined by $V_1(0) = 0$, which gives

$$P = \frac{\int_0^n v^\tau[\psi_{12}(0,\tau)(S\lambda_{23}(\tau)+b) + \psi_{11}(0,\tau)S\,\lambda_{13}(\tau)]d\tau}{\int_0^m v^\tau \psi_{11}(0,\tau)d\tau}. \quad \square$$

3.2 More complex models

The policy is still assumed to be described by a jump process X_t with finite state space $\mathcal{J} = \{1,\ldots,J\}$. We will consider examples such that the conditional expectation in (1.2) may depend on more information than the current state of X_t.

First we will study the immediate generalization of the case in Section 1, namely where a duration effect is put into the intensities and the payments: For each t, let $U_t \in \mathcal{R}_+$ measure the time spent in the current state X_t. We then obtain that $\tilde{X}_t = \{X_t, U_t\}$ becomes a Markov process with infinite state space $\mathcal{J} \times \mathcal{R}_+$.

Consider now a policy with the same terms as in Section 1, but with payments $b_i(t, U_t)$ and $b_{ij}(t, U_{t-})$ allowed to depend on the duration U_t in the visiting state. It is assumed that the mappings $(t,u) \to b_i(t,u)$ and $(t,u) \to b_{ij}(t,u)$ from $\mathcal{R}_+ \times \mathcal{R}_+$ into $\mathcal{R}$, are Borel measurable. The force of interest is still assumed to be constant, and the sequence $(T_n, Z_n)_{n\geq 0}$ has also the same meaning as in Section 1. The intensity process is now determined by

$$P(dN_{ij}(t) = 1 \,|\, \mathcal{F}^N_{t-}) = \lambda_{ij}(t, U_t)Y_i(t)dt, \;\; i \neq j,$$

where $(t,u) \to \lambda_{ij}(t,u)$ are deterministic functions.

The stochastic surplus V_0 over $[0,T]$, and $V_{(0,t]}$, $V_{(t,T]}$ are defined in the same way as in Section 1, simply by using $b_i(t,U_t)$ and $b_{ij}(t,U_{t-})$ instead of $b_i(t)$ and $b_{ij}(t)$ when the process is in state i at time t.

Similar to (1.1) we can write

$$V_0 = \sum_i \int_0^T v^s b_i(s,U_s)Y_i(s)ds + \sum_{i\neq j}\int_0^T v^s b_{ij}(s,U_{s-})dN_{ij}(s).$$

Similar to (1.3) we get

$$M_t = V_{(0,t]} + v^t V_{\tilde{X}_t}(t), \tag{2.1}$$

where $M_t = E(V_0 \,|\, \mathcal{F}_t^N)$. With the same arguments as in Section 1 we obtain similar to (1.6)

$$\begin{aligned} dV_{\tilde{X}_t}(t) &= \delta dt V_{\tilde{X}_t}(t) - \sum_i b_i(t, U_t) Y_i(t) dt \\ &\quad - \sum_{i \neq j} b_{ij}(t, U_{t-}) dN_{ij}(t) + v^{-t} dM_t. \end{aligned} \tag{2.2}$$

The representation theorem for M_t becomes:

Theorem 2.1 *Define martingales $M_{ij}(t)$ by*

$$dM_{ij}(t) = dN_{ij}(t) - \lambda_{ij}(t, U_t) Y_i(t) dt,$$

and introduce

$$R_{ij}(t, U_t) = b_{ij}(t, U_t) + V_{(j,0)}(t) - V_{(i,U_t)}(t).$$

The martingale $M_t = E(V_0 \,|\, \mathcal{F}_t^N)$ has the representation

$$M_t = E[V_0] + \int_0^t \sum_{i \neq j} v^s R_{ij}(s, U_{s-}) dM_{ij}(s),$$

whenever

$$\int_0^t \sum_{i \neq j} v^s |R_{ij}(s, U_s)| \lambda_{ij}(s, U_s) Y_i(s) ds < \infty, \quad a.s.$$

Proof: The technique is similar to the one used in Theorem 1.1.

Using $U_{T_n} = 0$, $\forall n \geq 0$, we obtain

$$\begin{aligned} M_{T_{n+1}} &= \sum_{k=1}^{n+1} \int_{T_{k-1}}^{T_k} v^s b_{Z_{k-1}}(s, s - T_{k-1}) ds + \sum_{k=1}^{n} v^{T_k} b_{Z_{k-1} Z_k}(T_k, T_k - T_{k-1}) \\ &\quad + v^{T_{n+1}} b_{Z_n Z_{n+1}}(T_{n+1}, T_{n+1} - T_n) + v^{T_{n+1}} V_{(Z_{n+1},0)}(T_{n+1}). \end{aligned}$$

The $\mathcal{F}^N_{T_n} \otimes \mathcal{B}_+$-measurable mappings $(\omega, t) \to f^{(n)}(\omega, t, j)$ become

$$\begin{aligned}
f^{(n)}(\omega, t, j) &= \sum_{k=1}^{n} \int_{T_{k-1}}^{T_k} v^s b_{Z_{k-1}}(s, s - T_{k-1}) ds + \int_{T_n}^{T_n+t} v^s b_{Z_n}(s, s - T_n) ds \\
&\quad + \sum_{k=1}^{n} v^{T_k} b_{Z_{k-1}Z_k}(T_k, T_k - T_{k-1}) + v^{T_n+t} b_{Z_n j}(T_n + t, t) \\
&\quad + v^{T_n+t} V_{(j,0)}(T_n + t).
\end{aligned}$$

The expression (1.9) for M_t on $T_n \leq t < T_{n+1}$ now modifies to

$$\begin{aligned}
h^{(n)}(t) &= \sum_{k=1}^{n} \int_{T_{k-1}}^{T_k} v^s b_{Z_{k-1}}(s, s - T_{k-1}) ds + \int_{T_n}^{t} v^s b_{Z_n}(s, s - T_n) ds \\
&\quad + \sum_{k=1}^{n} v^{T_k} b_{Z_{k-1}Z_k}(T_k, T_k - T_{k-1}) + v^t V_{(Z_n, t-T_n)}(t),
\end{aligned}$$

and the analogue of (1.14) is

$$\begin{aligned}
M_t - E[V_0] &= M_t - M_0 \\
&= \sum_{n=0}^{\infty} \int_0^t \sum_{j \neq Z_n} v^s [b_{Z_n j}(s, s - T_n) + V_{(j,0)}(s) - V_{(Z_n, s-T_n)}(s)] \\
&\quad \times I(T_n < s \leq T_{n+1})(dN_{Z_n j}(s) - \lambda_{Z_n j}(s, s - T_n) ds) \\
&= \int_0^t \sum_{i \neq j} v^s R_{ij}(s, U_{s-}) dM_{ij}(s), \qquad (2.3)
\end{aligned}$$

which ends the proof. □

We obtain the following theorem:

Theorem 2.2 *Over the continuity points of $b_i(t, u)$, $b_{ij}(t, u)$ and $\lambda_{ij}(t, u)$, the functions $t \to V_i(t, U_t)$ are continuously differentiable, and satisfy the system of differential equations:*

$$\begin{aligned}
\frac{dV_{(i,U_t)}}{dt}(t) &= \delta\, V_{(i,U_t)}(t) - b_i(t, U_t) \\
&\quad - \sum_{j \neq i} [b_{ij}(t, U_t) + V_{(j,0)}(t) - V_{(i,U_t)}(t)] \lambda_{ij}(t, U_t). \qquad (2.4)
\end{aligned}$$

Proof: Follows the same line as the proof of Theorem 1.2. Namely, using (2.2) and (2.3), the stochastic differential equation becomes

$$\begin{aligned} dV_{\tilde{X}_t}(t) &= \delta dt V_{\tilde{X}_t}(t) - \sum_i b_i(t,U_t)Y_i(t)dt - \sum_{i\in\mathcal{J}}\sum_{j\neq i} b_{ij}(t,U_{t-})dN_{ij}(t) \\ &\quad + \sum_{i\neq j} R_{ij}(t,U_{t-})dM_{ij}(t). \end{aligned}$$

Then over points where $X_t = X_{t-} = i$, we obtain

$$\begin{aligned} dV_{(i,U_t)}(t) &= \delta dt V_{(i,U_t)}(t) - b_i(t,U_t)dt \\ &\quad - \sum_{j\neq i}[b_{ij}(t,U_t) + V_{(j,0)}(t) - V_{(i,U_t)}(t)]\lambda_{ij}(t,U_t)dt, \end{aligned}$$

since $dN_{ij}(t) = 0$, for $j \neq i$, and $Y_i(t) = 1$, and $U_{t-} = U_t$. $\square$

If we assume that $V_{(i,u)}(t)$ have continuous partial derivatives for all $i \in \mathcal{J}$, denoted $\frac{\partial V_{(i,u)}}{\partial t}(t)$, $\frac{\partial V_{(i,u)}}{\partial u}(t)$, respectively, we obtain by definition

$$\frac{dV_{(i,U_t)}}{dt}(t) = \frac{\partial V_{(i,u)}}{\partial t}(t) + \frac{\partial V_{(i,u)}}{\partial u}(t),$$

since $dU_t = dt$ between the jums of X_t. Then (2.4) can also be cast as the partial differential equations

$$\begin{aligned} \frac{\partial V_{(i,u)}}{\partial t}(t) + \frac{\partial V_{(i,u)}}{\partial u}(t) &= \delta V_{(i,u)}(t) - b_i(t,u) \\ &\quad - \sum_{j\neq i}[b_{ij}(t,u) + V_{(j,0)}(t) - V_{(i,u)}(t)]\lambda_{ij}(t,u). \end{aligned}$$

We do not seem to gain much by introducing the partial derivatives, but what suffices is the system in (2.4) consisting of first order differential eqautions. This system is far more complicated than (1.15), since it involves the state reserves with duration zero. At least we can establish integral equations for $V_{(i,u)}(t),\ u \geq 0$: Using the initial condition $V_{(i,u)}(T) = a_i \in \mathcal{R}_+$, $i \in \mathcal{J}$, for all u, we obtain by (2.4)

$$\begin{aligned} V_{(i,U_t)}(t) &= \int_t^T e^{-\int_t^\tau(\delta+\lambda_i(s,U_s))ds} \\ &\quad \times\{b_i(\tau,U_\tau) + \sum_{j\neq i}[b_{ij}(\tau,U_\tau) + V_{(j,0)}(\tau)]\lambda_{ij}(\tau,U_\tau)\}d\tau \\ &\quad + a_i\, e^{-\int_t^T(\delta+\lambda_i(s,U_s))ds}. \end{aligned} \tag{2.5}$$

Since $U_\tau = U_t + \tau - t$ in (2.5), for $\tau \geq t$ the right hand side only depends on U_t. In particular we get for $U_t = 0$,

$$\begin{aligned} V_{(i,0)}(t) &= \int_t^T e^{-\int_t^\tau (\delta + \lambda_i(s,s-t))ds} \\ &\quad \times \{b_i(\tau, \tau - t) + \sum_{j \neq i} [b_{ij}(\tau, \tau - t) + V_{(j,0)}(\tau)] \lambda_{ij}(\tau, \tau - t)\} d\tau \\ &\quad + a_i \, e^{-\int_t^T (\delta + \lambda_i(s,s-t))ds}. \end{aligned} \tag{2.6}$$

The equations (2.5) and (2.6) could look differently if the amounts b_{ij} were dependent of the state reserves. If all the state reserves $V_{(i,0)}(t)$ have been evaluated, we can solve $V_{(i,u)}(t)$ from (2.5) for any fixed duration u. From (2.6) we can evaluate $V_{(i,0)}(t)$, but the terms involved seem to be complex functions in t, and an ordinary set of differential equations as (1.17) seems in general hard to establish. The integral equation system is of Volterra type, and must be solved numerically, see e.g. Berezin and Zhidkov (1965) or Baker (1977), but such numerical problems will not be discussed here. A way to obtain an ordinary set of differential equations in (2.6), is to assume a Markov structure on the intensities as in Section 1. An example of this could be the model in Example 1.1 combined with a qualifying period for disabled lives. This duration effect then makes the reserve for the disabled dependent of the past. The following example illustrates this.

Example 2.1. Qualifying period for disabled lives. Consider a policy of period n with terms as in Example 1.1, but modified such that there is a qualifying period of one year before receiving benefits as disabled. This gives with the model decription in Example 1.1 that $b_2(t, U_t) = b \, I(U_t \geq 1)$. The premium period is set to $m = n - 5$. The reserve for active state $V_1(t)$, does not depend on the duration, and the differential equations become

$$\frac{dW_1}{dt}(t) = -\lambda_{12}(t) W_{(2,0)}(t) + v^t [P - S \, \lambda_{13}(t)],$$

$$\frac{dW_{(2,U_t)}}{dt}(t) = \lambda_2(t) W_{(2,U_t)}(t) - \lambda_{21}(t) W_1(t) - v^t [S \, \lambda_{23}(t) + b \, I(U_t \geq 1)].$$

Thus with initial condition $V_1(n) = V_{(2,u)}(n) = 0$, for all u, we get

$$W_{(2,U_t)}(t) = \int_t^n e^{-\int_t^\tau \lambda_2(s) ds} \{\lambda_{21}(\tau) W_1(\tau) + v^\tau [S \lambda_{23}(\tau) + b \, I(U_\tau \geq 1)]\} d\tau.$$

For $U_t = u \geq 1$, $W_{(2,u)}(t)$ is independent of u, but for $0 \leq u \leq 1$

$$\begin{aligned} W_{(2,u)}(t) &= \int_t^n e^{-\int_t^\tau \lambda_2(s) ds} \{\lambda_{21}(\tau) W_1(\tau) + v^\tau S \lambda_{23}(\tau)\} d\tau \\ &\quad + b \int_{t+1-u}^n v^\tau e^{-\int_t^\tau \lambda_2(s)) ds} d\tau, \end{aligned} \tag{2.7}$$

where

$$b\int_{t+1-u}^{n} v^{\tau} e^{-\int_t^{\tau}\lambda_2(s)ds} d\tau = 0,$$

for $t \geq n-1+u$. Defining $b_2(t) = b\, e^{-\int_t^{t+1}(\delta+\lambda_2(s))ds}$ and differentiating $W_{(2,0)}(t)$, we get

$$\frac{dW_{(2,0)}}{dt}(t) = \lambda_2(t)W_{(2,0)}(t) - \lambda_{21}(t)W_1(t) - v^t[S\,\lambda_{23}(t) + b_2(t)]$$

where $b_2(t) = 0$, for $t \geq n-1$.

So, we arrive at a system of differential equations for $(W_1, W_{(2,0)})$ similar to the system in Example 1.1, now with

$$\mathbf{H}(t) = -v^t(S\,\lambda_{13}(t) - P, S\,\lambda_{23}(t) + b_2(t))'.$$

Let $\hat{\mathbf{\Phi}}$ be the basic solution to this system. Then with $\hat{\mathbf{\Phi}}^{-1} = (\psi_{ij})$ we get

$$\begin{aligned} V_1(t) \quad &= \quad \int_t^n v^{\tau-t}[\psi_{11}(t,\tau)S\lambda_{13}(\tau) + \psi_{12}(t,\tau)S\lambda_{23}(\tau)]d\tau \\ &\quad + \int_t^{n-1} v^{\tau-t}\psi_{12}(t,\tau)b_2(\tau)d\tau - P\int_t^m v^{\tau-t}\psi_{11}(t,\tau)d\tau, \end{aligned}$$

$$\begin{aligned} V_{(2,0)}(t) \quad &= \quad \int_t^n v^{\tau-t}[\psi_{21}(t,\tau)S\lambda_{13}(\tau) + \psi_{22}(t,\tau)S\lambda_{23}(\tau)]d\tau \\ &\quad + \int_t^{n-1} v^{\tau-t}\psi_{22}(t,\tau)b_2(\tau)d\tau - P\int_t^m v^{\tau-t}\psi_{21}(t,\tau)d\tau, \end{aligned}$$

where the integrals $\int_t^{n-1}$ and $\int_t^m$ are zero for $t \geq n-1$ and $t \geq m$, respectively. The equivalence premium is found as the solution to $V_1(0) = 0$.

For the numerical evaluation of $\hat{\mathbf{\Phi}}^{-1}(s,t)$ let $s = t_0 < t_1 < \cdots < t_n = t$ be a partition of the interval $[s,t]$ with division norm $\rho = \max_{1\leq k\leq n}\{(t_k - t_{k-1})\}$. Put $\Delta_k = t_k - t_{k-1},\ k = 1,\ldots,n$ and let $\mathbf{\Lambda}'$ be the matrix with elements that are the derivatives of the elements in $\mathbf{\Lambda}$. Because of the multiplicative structure on $\hat{\mathbf{\Phi}}^{-1}$ a second order Taylor expansion of the elements in $\hat{\mathbf{\Phi}}^{-1}$ gives that

$$\prod_{k=1}^{n}[\mathbf{1} - \mathbf{\Lambda}(t_{k-1})\Delta_k + \left(\mathbf{\Lambda}^2(t_{k-1}) - \mathbf{\Lambda}'(t_{k-1})\right)\frac{\Delta_k^2}{2}]$$

converges uniformly to $\hat{\mathbf{\Phi}}^{-1}(s,t)$ for $\rho \to 0$ with convergence rate of order ρ^2. This Taylor expansion is used as an approximation to $\hat{\mathbf{\Phi}}^{-1}$.

Then V_1 and $V_{(2,0)}$ are evaluated by the trapezoidal formula for integrals, and finally $V_{(2,u)}$ is evaluated by (2.7), also by the trapezoidal formula.

The following intensities have been used

$$\lambda_{12}(t) = \lambda_{21}(t) = 0.0004 + 10^{0.06t-5.46},$$

$$\lambda_{13}(t) = \lambda_{23}(t) = 0.0005 + 10^{0.038t-4.12}.$$

The intensities $\lambda_{12}(t)$, $\lambda_{13}(t)$ and $\lambda_{23}(t)$ above are used by Danish insurance companies today. To emphasize the necessity of a numerical procedure for the pair $(W_1, W_{(2,0)})$, a recovery intensity is used and, merely to illustrate chosen equal to $\lambda_{12}(t)$. The tables below show some results, where η denotes the annual interest rate.

Table 1: $S = 3000,\ b = 1000,\ \eta = 4.5\%$.

	$x = 40$ $n = 25$	$x = 50$ $n = 15$
P	63.8	104.8
$V_1(5)$	183.9	217.4
$V_{(2,0)}(5)$	11325.2	6605.0
$V_{(2,1/4)}(5)$	11564.3	6841.8
$V_1(10)$	331.8	368.7
$V_{(2,0)}(10)$	9183.7	3407.1
$V_{(2,1/2)}(10)$	9663.6	3879.9
$V_1(15)$	401.4	–
$V_{(2,0)}(15)$	6608.9	–
$V_{(2,3/4)}(15)$	7330.6	–

Table 2: $S = 3000,\ b = 1000,\ \eta = 10\%$.

	$x = 40$ $n = 25$	$x = 50$ $n = 15$
P	43.4	82.5
$V_1(5)$	127.2	177.8
$V_{(2,0)}(5)$	7494.4	5152.5
$V_{(2,1/4)}(5)$	7723.0	5378.9
$V_1(10)$	252.7	318.0
$V_{(2,0)}(10)$	6558.5	2946.9
$V_{(2,1/2)}(10)$	7020.2	3401.9
$V_1(15)$	333.7	–
$V_{(2,0)}(15)$	5155.7	–
$V_{(2,3/4)}(15)$	5854.7	–

Table 3: $S = 0,\ b = 1000,\ \eta = 10\%$.

	$x = 40$ $n = 25$	$x = 50$ $n = 15$
P	21.6	38.7
$V_1(5)$	59.2	57.7
$V_{(2,0)}(5)$	7284.1	4893.1
$V_{(2,1/4)}(5)$	7512.7	5119.4
$V_1(10)$	109.8	89.9
$V_{(2,0)}(10)$	6311.4	2736.8
$V_{(2,1/2)}(10)$	6773.2	3191.9
$V_1(15)$	125.5	–
$V_{(2,0)}(15)$	4894.4	–
$V_{(2,3/4)}(15)$	5593.4	–

To emphasize the variety of models that can be incorporated with the technique described here, we will as a final illustration extend the model above by allowing the intensities of X_t and the payment functions to depend on N_t the number of jumps occurred over the period $[0, t]$. More precisely, we assume that the total surplus over $[0, T]$ is given by

$$V_0 = \sum_i \int_0^T v^s b_i(s, U_s, N_s) Y_i(s) ds + \sum_{i \neq j} \int_0^T v^s b_{ij}(s, U_{s-}, N_{s-}) dN_{ij}(s),$$

and the intensities are determined by

$$P(dN_{ij}(t) = 1 \,|\, \mathcal{F}_{t-}^N) = \lambda_{ij}(t, U_t, N_t) Y_i(t) dt, \quad i \neq j,$$

where $(t, u) \to \lambda_{ij}(t, u, n)$ are deterministic functions. Similar to (2.1) we get

$$M_t = V_{(0,t]} + v^t V_{Q_t}(t),$$

where $Q_t = \{X_t, U_t, N_t\}$. The martingale $M_t = E(V_0 \,|\, \mathcal{F}_t^N)$ has now the representation

$$M_t = E[V_0] + \int_0^t \sum_{i \neq j} v^s R_{ij}(s, U_{s-}, N_{s-}) dM_{ij}(s),$$

whenever

$$\int_0^t \sum_{i \neq j} v^s |R_{ij}(s, U_s, N_s)| \lambda_{ij}(s, U_s, N_s) Y_i(s) ds < \infty, \quad a.s.,$$

where

$$dM_{ij}(t) = dN_{ij}(t) - \lambda_{ij}(t, U_t, N_t) Y_i(t) dt,$$

and

$$R_{ij}(t, U_t, N_t) = b_{ij}(t, U_t, N_t) + V_{(j,0,N_t+1)}(t) - V_{(i,U_t,N_t)}(t).$$

The analogue of (2.4) will then read:

$$\begin{aligned}\frac{dV_{(i,U_t,n)}}{dt}(t) &= \delta\, V_{(i,U_t,n)}(t) - b_i(t, U_t, n) \\ &\quad - \sum_{j \neq i} [b_{ij}(t, U_t, n) + V_{(j,0,n+1)}(t) - V_{(i,U_t,n)}(t)] \lambda_{ij}(t, U_t, n).\end{aligned}$$

We see that we obtain an infinite system of equations, and we shall not pursue how to treat this.

Chapter 4

The change of variable formula and applications

In this chapter, we will introduce some techniques which can be helpful for the stochastic analysis of insurance models. The chapter is basically a revised version of the ideas and applications studied in [22].

4.1 The change of variable formula

A real-valued, $\mathcal{F}_t$-adapted cadlag process $(Q_t)_{t\geq 0}$ is a process of finite variation, abbreviated FV, if the paths $t \to Q_t(\omega)$ are of bounded variation over finite intervals. An FV-process Q_t can always be decomposed into the sum of its continuous and discrete part as follows:

$$Q_t = Q_0 + Q_t^c + \sum_{0<s\leq t} \triangle Q_s,$$

where Q_t^c is the continuous part and $\triangle Q_t = Q_t - Q_{t-}$. The sum is of course well-defined since Q_t has finite variation.

Theorem 1.1 *(Change of variable formula) Let $Q_t = (Q_t^{(1)}, \ldots, Q_t^{(n)})$ be an n-tuple of FV-processes, and let $g : \mathcal{R}^n \to \mathcal{R}$ have continuous partial derivatives $\frac{\partial g}{\partial q_i}$, $i = 1, \ldots, n$. Then $(g(Q_t))_{t\geq 0}$ is an FV-process and*

$$g(Q_t) - g(Q_0) = \sum_{i=1}^{n} \int_0^t \frac{\partial g}{\partial q_i}(Q_s) dQ_s^{(i)\,c} + \sum_{0<s\leq t} \{g(Q_s) - g(Q_{s-})\}. \tag{1.1}$$

Let $Y_t = (Y_t^{(1)}, \ldots, Y_t^{(n)})$ be an n-tuple of FV-processes, such that Y_t jumps only at times $T_1 < T_2 < \ldots$ of a point process N_t. We associate a marked point process $N_t(A),\ A \in \mathcal{R}^n$, by letting the marks represent the increments of Y_t at the jumps. Since Y_t jumps only at the jump times of N_t, we can for any Borel function $h : \mathcal{R}^n \to \mathcal{R}$ write

$$\sum_{0<s\leq t} \{h(Y_s) - h(Y_{s-})\} = \int_0^t \int_{\mathcal{R}^n} \{h(Y_{s-} + y) - h(Y_{s-})\} dN_s(dy). \tag{1.2}$$

We have simply decomposed the sum according to the values of the increments of Y_t at the jumps.

Theorem 1.2 *Let $Y_t = (Y_t^{(1)}, \ldots, Y_t^{(n)})$ be an n-tuple of FV-processes satisfying (1.2) such that $N_t(A)$ admits an $\mathcal{F}_t$-intensity $\lambda_t(A)$, and let $g : \mathcal{R}^n \to \mathcal{R}$ be as in Theorem 1.1. Then the process*

$$\begin{aligned} M_t &= g(Y_t) - g(Y_0) - \sum_{i=1}^n \int_0^t \frac{\partial g}{\partial y_i}(Y_s) dY_s^{(i)\,c} \\ &\quad - \int_0^t \int_{\mathcal{R}^n} \{g(Y_s + y) - g(Y_s)\} \lambda_s(dy) ds, \end{aligned} \tag{1.3}$$

is a zero mean $\mathcal{F}_t$-martingale whenever

$$E\left[\int_0^t \int_{\mathcal{R}^n} |g(Y_s + y) - g(Y_s)| \lambda_s(dy) ds\right] < \infty.$$

Proof: Using (1.1) and (1.2), it is obvious that

$$M_t = \int_0^t \int_{\mathcal{R}^n} \{g(Y_{s-} + y) - g(Y_{s-})\}(dN_s(dy) - \lambda_s(dy)ds), \tag{1.4}$$

which is a zero mean $\mathcal{F}_t$-martingale since the integrand is predictable. Thus, in particular, the martingale is obtained by an integral representation. □

Consequently:

Theorem 1.3 *Assume that Y_t and g are as described in Theorem 1.2, but with the modification that $g(Y_t)$ is an $\mathcal{F}_t$-martingale. Then*

$$\sum_{i=1}^n \int_0^t \frac{\partial g}{\partial y_i}(Y_t) dY_t^{(i)\,c} + \int_0^t \int_{\mathcal{R}^n} \{g(Y_s + y) - g(Y_s)\} \lambda_s(dy) ds = 0. \tag{1.5}$$

Proof: By (1.3) we obtain that

$$\begin{aligned} g(Y_t) - g(Y_0) - M_t &= \sum_{i=1}^n \int_0^t \frac{\partial g}{\partial y_i}(Y_s) dY_s^{(i)\,c} \\ &\quad + \int_0^t \int_{\mathcal{R}^n} \{g(Y_s + y) - g(Y_s)\} \lambda_s(dy) ds, \end{aligned} \tag{1.6}$$

becomes a zero mean martingale and, therefore,

$$E\left[\sum_{i=1}^{n}\int_s^t \frac{\partial g}{\partial y_i}(Y_u)dY_u^{(i)\,c} + \int_s^t \int_{\mathcal{R}^n} \{g(Y_u+y)-g(Y_u)\}\lambda_u(dy)du \,|\, \mathcal{F}_s^N\right] = 0, \;\; \forall\, s<t.$$

Since the expectation concerns an FV-process with continuous paths, the martingale is constant and hence zero, see e.g. Chung and Williams (1990, pp. 87-88). □

4.2 Some examples

Example 2.1. Thiele's differential equation. We will illustrate, how we can use Theorem 1.3 to arrive at the martingale representations presented in Chapter 3, which was the tool for obtaining the differential equations for the state reserves. We will only consider the pure Markov model, and below we make use of the notation of Chapter 3.

Assume that $V_{X_t}(t)$ is an FV-process, and decompose $v^t V_{X_t}(t)$ into its continuous and discrete part, which reads

$$v^t V_{X_t}(t) = V_{X_0}(0) + v^t V_{X_t}^c(t) + \sum_{0<s\leq t} v^s \{V_{X_s}(s) - V_{X_{s-}}(s-)\},$$

where $V_{X_t}^c(t)$ is the continuous part of $V_{X_t}(t)$. By assuming that $V_{X_t}(t)$ jumps only at the jump times of X_t, we can similar to (1.2) write

$$\sum_{0<s\leq t} v^s \{V_{X_s}(s) - V_{X_{s-}}(s-)\} = \sum_{i\neq j}\int_0^t v^s\{V_j(s)-V_i(s)\}dN_{ij}(s).$$

For simplicity assume that $V_{X_0}(0) = 0$. Relation (1.3) in Chapter 3 then reads

$$\begin{aligned}
M_t &= V_{(0,t]} + v^t V_{X_t}(t) \\
&= V_{(0,t]} + v^t V^c_{X_t}(t) + \sum_{i \neq j} \int_0^t v^s \{V_j(s) - V_i(s)\} dN_{ij}(s) \\
&= \sum_i \int_0^t v^s b_i(s) Y_i(s) ds + v^t V^c_{X_t}(t) \\
&\quad + \sum_{i \neq j} \int_0^t v^s \{b_{ij}(s) + V_j(s) - V_i(s)\} dN_{ij}(s) \\
&= \sum_i \int_0^t v^s b_i(s) Y_i(s) ds + v^t V^c_{X_t}(t) \\
&\quad + \sum_{i \neq j} \int_0^t v^s \{b_{ij}(s) + V_j(s) - V_i(s)\} \lambda_{ij}(s) Y_i(s) ds + M^*_t,
\end{aligned}$$

where

$$M^*_t = \sum_{i \neq j} \int_0^t v^s \{b_{ij}(s) + V_j(s) - V_i(s)\}(dN_{ij}(s) - \lambda_{ij}(s) Y_i(s) ds),$$

is a zero mean martingale under the same assumptions as in Theorem 1.1, Chapter 3. Thus

$$\begin{aligned}
M_t - M^*_t &= \sum_i \int_0^t v^s b_i(s) Y_i(s) ds + v^t V^c_{X_t}(t) \\
&\quad + \sum_{i \neq j} \int_0^t v^s \{b_{ij}(s) + V_j(s) - V_i(s)\} \lambda_{ij}(s) Y_i(s) ds
\end{aligned}$$

becomes a zero mean martingale, and is obviously continuous and of bounded variation, and hence identically zero. We have then obtained the representation of M_t in Theorem 1.1, Chapter 3. □

Example 2.2. The distribution of jump processes. We will study how to establish an integro-differential equation for the function

$$F(t, x) = P(X^{(f)}_t \leq x), \quad x \in \mathcal{R}.$$

The technique is only applicable when $N_t(A)$ are Poisson processes, implying that the intensity is deterministic. The idea is to consider the process $t \to I(X^{(f)}_t \leq x)$, which is cadlag and purely discrete. Using (1.1) and (1.2) we obtain

$$\begin{aligned}
& I(X_t^{(f)} \leq x) - I(x \geq 0) \\
&= \sum_{0<s\leq t} \{I(X_s^{(f)} \leq x) - I(X_{s-}^{(f)} \leq x)\} \\
&= \int_0^t \int_{\mathcal{Z}} \{I(X_{s-}^{(f)} \leq x - f(s,z)) - I(X_{s-}^{(f)} \leq x)\} dN_s(dz).
\end{aligned}$$

Taking the mean and using that the intensity is deterministic, we get

$$\begin{aligned}
& F(t,x) - I(x \geq 0) \\
&= E\left[\int_0^t \int_{\mathcal{Z}} \{I(X_{s-}^{(f)} \leq x - f(s,z)) - I(X_{s-}^{(f)} \leq x)\} dN_s(dz)\right] \\
&= E\left[\int_0^t \int_{\mathcal{Z}} \{I(X_{s-}^{(f)} \leq x - f(s,z)) - I(X_{s-}^{(f)} \leq x)\} \lambda_s(dz) ds\right] \\
&= \int_0^t \int_{\mathcal{Z}} \{F(s, x - f(s,z)) - F(s,x)\} \lambda_s G_s(dz) ds.
\end{aligned}$$

So, over the continuity points of $\lambda_t(A)$ and $f(t,z)$, we obtain by differentiation the following integro-differential equation:

$$\frac{dF}{dt}(t,x) = -\lambda_t F(t,x) + \lambda_t \int_{\mathcal{Z}} F(t, x - f(t,z)) G_t(dz),$$

where F satisfies the initial condition $F(0,x) = I(x \geq 0)$. If f takes only positive values, the domain of integration is reduced to $\{z | x \geq f(t,z)\}$, which is convenient in a numerical implementation. We refer to [25] for some numerical implementations and for a more refined treatment on the subject based on a martingale approach. □

Example 2.3. Moments of jump processes. We assume that the $N_t(A)$ are Poisson processes, and will obtain expressions for the central moments

$$\mu_t^{(k)} = E(X_t^{(f)} - E[X_t^{(f)}])^k, \quad k = 2, 3, 4.$$

Assume in the following that

$$\int_0^t \int_{\mathcal{Z}} |f^k(s,z)| \lambda_s(dz) ds < \infty, \quad k = 2, 3, 4.$$

By the definition of intensity, we immediately obtain that $\mu_t = E[X_t^{(f)}]$ is given as

$$\mu_t = \int_0^t \int_{\mathcal{Z}} f(s,z) \lambda_s(dz).$$

This is also a natural consequence of the martingale property (0.6) in Chapter 2. It should therefore be well-known that

$$\mu_t^{(2)} = \int_0^t \int_{\mathcal{Z}} f^2(s,z)\lambda_s(dz)ds. \tag{2.7}$$

We will also show this in light of the change of variable formula and use this technique to express the third and fourth order central moments. Define

$$Y_t = X_t^{(f)} - E[X_t^{(f)}],$$

which obviously is an FV-process with $Y_t^c = -E[X_t^{(f)}]$. Using (1.1) with $g(y) = y^2$, we obtain

$$Y_t^2 = -2\int_0^t Y_s \int_{\mathcal{Z}} f(s,z)\lambda_s(dz)ds + \sum_{0<s\leq t} \{Y_s^2 - Y_{s-}^2\}.$$

Furthermore,

$$\begin{aligned} Y_t^2 - Y_{t-}^2 &= (Y_{t-} + \int_{\mathcal{Z}} f(t,z)dN_t(dz))^2 - Y_{t-}^2 \\ &= 2\,Y_{t-}\int_{\mathcal{Z}} f(t,z)dN_t(dz) + \int_{\mathcal{Z}} f^2(t,z)dN_t(dz). \end{aligned}$$

We gather

$$\begin{aligned} Y_t^2 &= 2\int_0^t Y_{s-}\int_{\mathcal{Z}} f(s,z)(dN_s(dz) - \lambda_s(dz)ds) \\ &\quad + \int_0^t \int_{\mathcal{Z}} f^2(s,z)dN_s(dz), \end{aligned}$$

and finally obtain (2.7) by taking the mean. We proceed similarly for the third and fourth order central moment:

$$Y_t^3 = -3\int_0^t Y_s^2 \int_{\mathcal{Z}} f(s,z)\lambda_s(dz)ds + \sum_{0<s\leq t} \{Y_s^3 - Y_{s-}^3\}.$$

As above we can write

$$\begin{aligned} Y_t^3 - Y_{t-}^3 &= (Y_{t-} + \int_{\mathcal{Z}} f(t,z)dN_t(dz))^3 - Y_{t-}^3 \\ &= 3\,Y_{t-}\int_{\mathcal{Z}} f^2(t,z)dN_t(dz) + 3\,Y_{t-}^2\int_{\mathcal{Z}} f(t,z)dN_t(dz) \\ &\quad + \int_{\mathcal{Z}} f^3(t,z)dN_t(dz), \end{aligned}$$

and obtain

$$\begin{aligned} Y_t^3 &= 3\int_0^t Y_{s-}^2 \int_{\mathcal{Z}} f(s,z)(dN_s(dz) - \lambda_s(dz)ds) \\ &\quad +3\int_0^t Y_{s-} \int_{\mathcal{Z}} f^2(s,z)dN_s(dz) + \int_0^t \int_{\mathcal{Z}} f^3(s,z)dN_s(dz). \end{aligned}$$

The first term has mean zero (it is a zero mean martingale), and so has also the second, since $\lambda_t(A)$ is deterministic and $E[Y_t] = 0$. We conclude that

$$\mu_t^{(3)} = \int_0^t \int_{\mathcal{Z}} f^3(s,z)\lambda_s(dz)ds. \tag{2.8}$$

Finally,

$$Y_t^4 = -4\int_0^t Y_s^3 \int_{\mathcal{Z}} f(s,z)\lambda_s(dz)ds + \sum_{0<s\le t} \{Y_s^4 - Y_{s-}^4\},$$

and

$$\begin{aligned} Y_t^4 - Y_{t-}^4 &= (Y_{t-} + \int_{\mathcal{Z}} f(t,z)dN_t(dz))^4 - Y_{t-}^4 \\ &= 4\,Y_{t-}\int_{\mathcal{Z}} f^3(t,z)dN_t(dz) + 4\,Y_{t-}^3 \int_{\mathcal{Z}} f(t,z)dN_t(dz) \\ &\quad +6\,Y_{t-}^2 \int_{\mathcal{Z}} f^2(t,z)dN_t(dz) + \int_{\mathcal{Z}} f^4(t,z)dN_t(dz). \end{aligned}$$

Using arguments similar to those leading to (2.8), we arrive at

$$\begin{aligned} \mu_t^{(4)} &= \int_0^t \int_{\mathcal{Z}} f^4(s,z)\lambda_s(dz)ds + 6\int_0^t \mu_s^{(2)} \int_{\mathcal{Z}} f^2(s,z)\lambda_s(dz)ds \qquad (2.9) \\ &= \int_0^t \int_{\mathcal{Z}} f^4(s,z)\lambda_s(dz)ds + 6\int_0^t \mu_s^{(2)} d\mu_s^{(2)} \\ &= \int_0^t \int_{\mathcal{Z}} f^4(s,z)\lambda_s(dz)ds + 3(\mu_t^{(2)})^2. \qquad (2.10) \end{aligned}$$

For instance, we can use these expressions to evaluate higher order moments of the discounted risk process

$$\tilde{X}_t = \sum_{i=1}^{N_t} e^{-\delta T_i} Z_i, \quad Z_i \in \mathcal{R},$$

where δ is the constant force of interest. $\square$

Example 2.4. Exponential martingales. We will derive some well-known exponential martingales, which we will use in connection with diffusion approximations in Chapter 6. We will prove that the process

$$M_t^{(f)} = e^{X_t^{(f)} - \int_0^t \int_{\mathcal{Z}} (e^{f(s,z)} - 1)\lambda_s(dz)ds}, \tag{2.11}$$

is an $\mathcal{F}_t$-martingale with mean one whenever

$$E[e^{X_t^{(f)} - \int_0^t \int_{\mathcal{Z}} (e^{f(s,z)} - 1)\lambda_s(dz)ds}] < \infty.$$

Using the change of variable formula with

$$Y_t = X_t^{(f)} - \int_0^t \int_{\mathcal{Z}} (e^{f(s,z)} - 1)\lambda_s(dz)ds,$$

and $g(y) = e^y$, we obtain

$$\begin{aligned} M_t^{(f)} &= 1 - \int_0^t e^{Y_s} \int_{\mathcal{Z}} (e^{f(s,z)} - 1)\lambda_s(dz)ds + \sum_{0<s\le t} \{e^{Y_s} - e^{Y_{s-}}\} \\ &= 1 - \int_0^t e^{Y_s} \int_{\mathcal{Z}} (e^{f(s,z)} - 1)\lambda_s(dz)ds \\ &\quad + \int_0^t \int_{\mathcal{Z}} e^{Y_{s-}} (e^{f(s,z)} - 1)dN_s(dz) \\ &= \int_0^t \int_{\mathcal{Z}} e^{Y_{s-}} (e^{f(s,z)} - 1)(dN_s(dz) - \lambda_s(dz)ds), \end{aligned}$$

which is an $\mathcal{F}_t$-martingale since the integrand is $\mathcal{F}_t$-predictable.

Similarly, we can arrive at the complex valued martingale

$$M_t^{(f)} = e^{iuX_t^{(f)} - \int_0^t \int_{\mathcal{Z}} (e^{iuf(s,z)} - 1)\lambda_s(dz)ds},$$

where i is the imaginary unit ($i^2 = -1$) and $u \in \mathcal{R}$. The process

$$\phi_t(u) = e^{\int_0^t \int_{\mathcal{R}_+} (e^{iuf(s,z)} - 1)\lambda_s(dz)ds}$$

is called the characteristic function of $X_t^{(f)}$ and characterizes its distribution. Assume e.g. that $\lambda_t(A)$ is deterministic. Then $X_t^{(f)}$ has independent increments since the martingale property yields

$$E[e^{iu(X_t^{(f)} - X_s^{(f)})} \mid \mathcal{F}_s^N] = e^{\int_s^t \int_{\mathcal{Z}} (e^{iuf(\tau,z)} - 1)\lambda_\tau(dz)d\tau}.$$

See e.g. Delbaen and Haezendonck (1987) for a special case. $\square$

Chapter 5

Stochastic differential equations for ruin probabilities

In this chapter we will focus on differential equations for ruin probabilities. The aim is to understand how one can establish differential equations for quite general models and furthermore, how these generalizations are related to the cases in the references listed. The approach introduced here seems to be new.

We start by defining a stochastic process of conditional ruin probabilities where time varies over an interval $[0, T)$. More precisely, we will be interested in e.g. the probability of ruin over (t, T) given the reserve at time t. These conditional probabilities lead to a process, and the key point is that this process, stopped at the time of ruin, becomes a martingale. Using this property, we can obtain (stochastic) differential equations for the process of conditional ruin probabilities, by use of the martingale representation theorem.

Section 1 treats the Markov case. For simplicity, we base the mathematical steps on the change of variable formula, but as in Chapter 3, it is the representation of the martingale which is essential to arrive at the differential equations, see the comments following (1.17). Also we give some numerical examples to obtain the probability of ruin over a finite time period.

In Section 2 we proceed to more complex models, where e.g. the premium or the claims intensity can fluctuate in such a way that the risk model is no longer Markov, but such that Markovization is still possible. First we study a case where the intensity depends on the history only via the number of jumps and, second, we study the risk model in a Markovian environment.

5.1 The risk reserve as a Markov process

5.1.1 The martingale approach

We start with the following definition of the risk reserve at time t:

$$R_t = R_0 + \int_0^t b(s, R_s)ds - \int_0^t \int_{\mathcal{Z}} f(s,z)dN_s(dz). \tag{1.1}$$

The mapping $(t,r) \to b(t,r)$ from $\mathcal{R}_+ \times \mathcal{R}$ to $\mathcal{R}$ is assumed to be piecewise continuous in t and r, and could represent premium income or annuity payments to the insured. The risk reserve is a cadlag process (right continuous paths with left limits), and it is required throughout that R_t is $\mathcal{F}_t$-measurable.

The time of ruin is defined as

$$\tau = \inf\{t \geq 0 \,|\, R_t < 0\},$$

which is an $\mathcal{F}_t$-stopping time, that is, $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. Since we allow $b(t,r)$ to take negative values, ruin is not only caused by a jump. Ruin between jumps could e.g. happen if the company pays out pensions, or if it invests its reserve and gets a negative outcome of its investments. To prevent technical details from obscuring the presentation, we omit the possibility of payments at discrete times such as for instance lump sum payments governed by the size of R_t.

We assume that $N_t(A)$ is a Poisson process for each A, and the interpretation of G_t in (0.4), Chapter 2, becomes then

$$\int_A G_t(dz) = P(Z_n \in A \,|\, T_n = t). \ \forall\, n.$$

Then, since $b(t, R_t)$ depends on the history only through R_t, the reserve R_t becomes an $\mathcal{F}_t$-Markov process (it need not have independent increments), that is $\sigma(R_s,\, s \geq t)$ and $\mathcal{F}_t$ are independent given R_t. Also, we could allow f to depend on R_{t-}, since this would not destroy the Markov property of R_t and the mathematical steps leading to Theorem 1.1 below. For notational convenience we omit this.

For a fixed $T \leq \infty$ we write

$$\begin{aligned} I(\tau < T) &= I(\tau \leq t) + I(t < \tau < T) \\ &= I(\tau \leq t) + I(\tau > t) I(\inf_{t \leq s < T} R_s < 0) \\ &= I(\tau < t) + I(\tau \geq t) I(\inf_{t \leq s < T} R_s < 0). \end{aligned} \tag{1.2}$$

Defining $M_t = P(\tau < T \,|\, \mathcal{F}_t) = E(I(\tau < T) \,|\, \mathcal{F}_t)$, taking conditional expectation w.r.t. $\mathcal{F}_t$ in (1.2), and using the Markov property, we get

$$\begin{aligned} M_t &= I(\tau \le t) + P(t < \tau < T \,|\, \mathcal{F}_t) \\ &= I(\tau \le t) + I(\tau > t) P(\inf_{t \le s < T} R_s < 0 \,|\, \mathcal{F}_t) \\ &= I(\tau \le t) + I(\tau > t)\Psi(t, R_t) \\ &= I(\tau < t) + I(\tau \ge t)\Psi(t, R_t), \end{aligned} \tag{1.3}$$

where the function $\Psi : \mathcal{R}_+ \times \mathcal{R} \to [0,1]$ is

$$\Psi(t,r) = P(\inf_{t \le s < T} R_s < 0 \,|\, R_t = r), \tag{1.4}$$

the probability of ruin after time t with reserve r at time t. Also, we get

$$\Psi(t,r) = 1, \;\; r < 0. \tag{1.5}$$

If we assume that $\Psi(t,r)$ is continuous in t, r, we obtain by (1.3) that M_t is right-continuous with left-hand limits given by

$$M_{t-} = I(\tau < t) + I(\tau \ge t)\Psi(t, R_{t-}). \tag{1.6}$$

In the sequel, we chose the version of Ψ such that M_t is cadlag.

Inserting $t \wedge \tau$ in (1.3), we get

$$M_{t\wedge\tau} = \Psi(t \wedge \tau, R_{t\wedge\tau}), \;\; t \in [0, T), \tag{1.7}$$

which in particular gives that $\Psi(t\wedge\tau, R_{t\wedge\tau})$ is a (uniformly integrable) martingale, and using this idea we will derive a differential equation for the non-ruin probability

$$\Phi(t, R_t) = 1 - \Psi(t, R_t).$$

As outlined in the introduction, we see that the martingale property in (1.7) is derived using only the Markov property, and not the particular functional structure of R_t.

The function $\Psi(t,r)$ is independent of t if e.g. R_t is a homogeneous Markov process and $T = \infty$, and becomes then identical to the probability of ruin in infinite time with initial reserve r. With a finite time horizon or e.g. time dependent intensities the dependence on t cannot in general be suppressed. In the numerical procedures for evaluating ruin probabilities, we primarily operate with $T < \infty$

since it is then possible to state the initial condition $\Phi(T,r)=1$ for all r, and then solve $\Phi(t,r)$ over $[0,T)$ for some values of r.

Another relation: Using that $M_{t\wedge\tau}=E(I(\tau<T\,|\,\mathcal{F}_{t\wedge\tau})$ (optional sampling), we obtain by taking conditional expectation on both sides in (1.7) w.r.t. the $\mathcal{F}_{t\wedge\tau}$-measurable stochastic variable $(t\wedge\tau, R_{t\wedge\tau})$, that

$$P(\tau<T\,|\,t\wedge\tau, R_{t\wedge\tau})=\Psi(t\wedge\tau, R_{t\wedge\tau}). \tag{1.8}$$

We now introduce the technique that leads to the differential equations. We assume that $\Phi(t,r)$ has continuous partial derivatives for $t,r>0$, which are denoted $\frac{\partial\Phi}{\partial t}(t,r)$ and $\frac{\partial\Phi}{\partial r}(t,r)$, respectively. The change of variable formula yields for $t\in[0,T)$:

$$\begin{aligned}
&\Phi(t\wedge\tau, R_{t\wedge\tau})-\Phi(0,R_0)\\
&= \int_0^{t\wedge\tau}\frac{\partial\Phi}{\partial t}(s,R_s)ds+\int_0^{t\wedge\tau}\frac{\partial\Phi}{\partial r}(s,R_s)b(s,R_s)ds\\
&\quad+\sum_{s\le t\wedge\tau}[\Phi(s,R_s)-\Phi(s,R_{s-})]\\
&= \int_0^{t\wedge\tau}\frac{\partial\Phi}{\partial t}(s,R_s)ds+\int_0^{t\wedge\tau}\frac{\partial\Phi}{\partial r}(s,R_s)b(s,R_s)ds\\
&\quad+\int_0^{t\wedge\tau}\int_{\mathcal{Z}}[\Phi(s,R_{s-}-f(s,z))-\Phi(s,R_{s-})]dN_s(dz)\\
&= \int_0^{t\wedge\tau}\frac{\partial\Phi}{\partial t}(s,R_s)ds+\int_0^{t\wedge\tau}\frac{\partial\Phi}{\partial r}(s,R_s)b(s,R_s)ds\\
&\quad+\int_0^{t\wedge\tau}\int_{\mathcal{Z}}[\Phi(s,R_s-f(s,z))-\Phi(s,R_s)]\lambda_s(dz)ds+M_t^*,
\end{aligned} \tag{1.9}$$

where

$$M_t^*=\int_0^t\int_{\mathcal{Z}}I(\tau\ge s)[\Phi(s,R_{s-}-f(s,z))-\Phi(s,R_{s-})]dM_s(dz), \tag{1.10}$$

and $M_t(A)$ is given by (0.5) in Chapter 2, and we have replaced R_{t-} with R_t when integration is w.r.t. Lebesgue measure. Since R_{t-} and $I(\tau\ge t)$ are processes with left-continuous paths and f is deterministic, the integrand in (1.10) is $\mathcal{F}_t$-

predictable, hence M_t^* is a zero mean $\mathcal{F}_t$-martingale. Thus

$$\begin{aligned}
&\Phi(t\wedge\tau, R_{t\wedge\tau}) - \Phi(0,R_0) - M_t^* \\
&= \int_0^{t\wedge\tau} \frac{\partial\Phi}{\partial t}(s,R_s)ds + \int_0^{t\wedge\tau} \frac{\partial\Phi}{\partial r}(s,R_s)b(s,R_s)ds - \int_0^{t\wedge\tau} \Phi(s,R_s)\lambda_s ds \\
&\quad + \int_0^{t\wedge\tau}\int_{\mathcal{Z}} \Phi(s, R_s - f(s,z))\lambda_s(dz)ds \qquad (1.11)
\end{aligned}$$

is a zero mean $\mathcal{F}_t$-martingale. Obviously (1.11) is a continuous FV-process and then, by Theorem 1.3 in Chapter 4, zero. Thus

$$\begin{aligned}
&\int_0^{t\wedge\tau} \frac{\partial\Phi}{\partial t}(s,R_s)ds + \int_0^{t\wedge\tau} \frac{\partial\Phi}{\partial r}(s,R_s)b(s,R_s)ds \\
&= \int_0^{t\wedge\tau} \Phi(s,R_s)\lambda_s ds - \int_0^{t\wedge\tau}\int_{\mathcal{Z}} \Phi(s, R_s - f(s,z))\lambda_s(dz)ds. \qquad (1.12)
\end{aligned}$$

Using (1.5) ($\Phi(t,R_t) = 0,\ R_t < 0$), we can always replace the double integral $\int_0^t \int_{\mathcal{Z}}$ with $\int_0^t \int_{\{z|\, R_s \geq f(s,z)\}}$.

We need the more generel formulation: Fix an arbitrary $t' < T$, and define

$$\tau' = \inf\{t \geq t' \mid R_t < 0\},$$

which is the first time of ruin after time t', and repeat the arguments leading to (1.7) to state that

$$M_t' = \Phi(t\wedge\tau', R_{t\wedge\tau'}), \quad t \in [t',T),$$

is an $\mathcal{F}_t$-martingale. Similarly to (1.12), we can then arrive at

$$\begin{aligned}
&\int_{t'}^{t\wedge\tau'} \frac{\partial\Phi}{\partial t}(s,R_s)ds + \int_{t'}^{t\wedge\tau'} \frac{\partial\Phi}{\partial r}(s,R_s)b(s,R_s)ds \\
&= \int_{t'}^{t\wedge\tau'} \Phi(s,R_s)\lambda_s ds - \int_{t'}^{t\wedge\tau'}\int_{\mathcal{Z}} \Phi(s, R_s - f(s,z))\lambda_s(dz)ds. \qquad (1.13)
\end{aligned}$$

Theorem 1.1 *For any fixed $t' \in [0,T)$, the process $\Psi(t\wedge\tau', R_{t\wedge\tau'})$, $t \in [t',T)$, is a (uniformly integrable) martingale, and over the continuity points of $\lambda_t(A)$ and f,* *the function $\Phi(t,r) = 1 - \Psi(t,r)$ satisfies the partial integro-differential equation*

$$\frac{\partial \Phi}{\partial t}(t,r) + \frac{\partial \Phi}{\partial r}(t,r)b(t,r)$$

$$= \Phi(t,r)\lambda_t - \int_{\{z \,|\, r \geq f(t,z)\}} \Phi(t, r - f(t,z))\lambda_t(dz), \quad t \in (0,T), \ r > 0. \tag{1.14}$$

Proof: Follows by (1.13) since t' is arbitrarily chosen on $[0,T)$ and by definition $\tau' \geq t'$. $\square$

We get by (1.14) that $\Phi(t,R_t)$, for $R_t > 0$, satisfies $\tilde{\mathcal{A}}\Phi = 0$, where $\tilde{\mathcal{A}}$ is the extended generator of R_t, but remark that $\Phi(t,R_t)$ is not a martingale.

In the case where $b(t,r)$ is independent of r, which we denote $b(t)$, the R_t process in (1.1) has independent increments and therefore, for $b(t) > 0$ (ruin only at jumps), we can for $t \in [0,T], \ u \geq 0$, heuristically establish the relation

$$\Psi(t,u) = \int_t^T \int_{\mathcal{Z}} e^{-\int_t^\eta \lambda_s ds} \Psi(\eta, u + \int_t^\eta b(s)ds - f(\eta,z))\lambda_\eta(dz)d\eta.$$

By differentiation we get for $t \in (0,T), \ u > 0$

$$\begin{aligned}
\frac{\partial \Psi}{\partial t}(t,u) &= -\int_{\mathcal{Z}} \Psi(t, u - f(t,z))\lambda_t(dz) + \lambda_t \Psi(t,u) \\
&\quad + \int_t^T \int_{\mathcal{Z}} e^{-\int_t^\eta \lambda_s ds} \frac{\partial \Psi}{\partial t}(\eta, u + \int_t^\eta b(s)ds - f(\eta,z))\lambda_\eta(dz)d\eta \\
&= \lambda_t \Psi(t,u) - \int_{\mathcal{Z}} \Psi(t, u - f(t,z))\lambda_t(dz) - b(t)\frac{\partial \Psi}{\partial r}(t,u).
\end{aligned}$$

which leads to (1.14). In general this approach is, of course, not applicable.

When evaluating (1.14) numerically it is convenient to transform it into a first order integro-differential equation. Firstly, let U_t be a non-negative function satisfying the differential equation

$$\frac{dU_t}{dt} = b(t, U_t), \tag{1.15}$$

meaning that U_t plays the role of R_t between the jumps. Using integration by parts on $\Phi(t,U_t)$, we can then use (1.14) to establish the integro-differential equation

$$d\Phi(t,U_t) = \Phi(t,U_t)\lambda_t dt - \int_{\{z \,|\, U_t \geq f(t,z)\}} \Phi(t, U_t - f(t,z))\lambda_t(dz)dt. \tag{1.16}$$

Furthermore, introduce the function $\tilde{\Phi}_u(t) \equiv \Phi(t, u + \int_0^t b(s, U_s))$, $u \in \mathcal{R}$, where $U_t = u + \int_0^t b(s, U_s)ds$, is referred to as the characteristic curve or just the characteristic, see e.g. Smith (1985, pp. 175-181). Using (1.15) and (1.16), we can then obtain the differential equation

$$\frac{d\tilde{\Phi}_u}{dt}(t) = \tilde{\Phi}_u(t)\lambda_t - \int_{\{z \,|\, U_t \geq f(t,z)\}} \tilde{\Phi}_{u-f(t,z)}(t)\lambda_t(dz). \tag{1.17}$$

With the initial condition $\tilde{\Phi}_u(T) = 1$, $u \geq -\int_0^T b(s, U_s)ds$, and $T < \infty$, (1.17) must be solved numerically, see Subsection 1.3 for an example.

When operating with point process histories we could more directly derive (1.16) by finding the martingale representation of $\Phi(t \wedge \tau, R_{t\wedge\tau})$ similarly as described in Theorem 1.1 and 2.1 in Chapter 3. Combining (1.10) and the fact that the martingale in (1.11) is zero, we already know that the representation must have the form

$$\begin{aligned} &\Phi(t \wedge \tau, R_{t\wedge\tau}) - \Phi(0, R_0) \\ &= \int_0^{t\wedge\tau} \int_{\mathcal{Z}} [\Phi(s, R_{s-} - f(s,z)) - \Phi(s, R_{s-})] dM_s(dz), \quad t \in [0, T], \end{aligned}$$

or more generally

$$\begin{aligned} &\Phi(t \wedge \tau', R_{t\wedge\tau'}) - \Phi(t', R_{t'}) \\ &= \int_{t'}^{t\wedge\tau'} [\Phi(s, R_{s-} - f(s,z)) - \Phi(s, R_{s-})] dM_s(dz), \quad t \in [t', T]. \end{aligned}$$

The representation states that $\Phi(t, R_t)$ between the jumps ($dN_t = 0$) develops in accordance with the differential equation (1.16), where R_t takes the role of U_t. Therefore, using the representation theorem, we can arrive directly at (1.16) and then (1.17), which is the essential equation in a numerical procedure. So it is not crucial to assume that Φ has continuous partial derivatives (governed by a vector field), which is required when operating with generators for PD Markov processes. For instance, we get by (1.16) that $t \to \Phi(t, U_t)$ is (absolutely) continuous and, over the continuity points of $\lambda_t(A)$ and f, is a differentiable function satisfying (1.17). In this paper integration by parts is chosen for convenience only, and is applicable when considering processes which can be transformed into Markov processes. On the other hand, it seems more informative to go via the representation theorem.

In the following we focus on the homogeneous case. This is obtained by assuming that $b(t, r)$, $f(t, z)$, and $\lambda_t(A)$ are independent of t. The intensity is then given by

$$\lambda_t(dz) = \lambda\, G(dz), \tag{1.18}$$

where $\lambda\,(>0)$ is the intensity of the homogeneous Poisson process N_t, and G is the distribution of the i.i.d. (independent and identically distributed) random variables $Z_1, Z_2, \ldots$ We consider the reserve

$$R_t = R_0 + \int_0^t b(R_s)ds - \int_0^t \int_{\mathcal{R}} z\, dN_s(dz), \tag{1.19}$$

where in particular $\int_0^t \int_{\mathcal{R}} z\, dN_s(dz) = \sum_{i=1}^{N_t} Z_i$, is a compound Poisson process. Using the property of homogeneity, we get

$$\begin{aligned}
\Psi(t,r) &= P(\inf_{t\le s<T} R_s < 0 \,|\, R_t = r) \\
&= P(\inf_{0\le s<T-t} R_s < 0 \,|\, R_0 = r) \\
&= P(\tau < T-t \,|\, R_0 = r).
\end{aligned}$$

We can then as well consider $\Phi^* = 1 - \Psi^*$, with

$$\Psi^*(t,r) = P(\tau < t \,|\, R_0 = r),$$

satisfying $\Psi^*(0,r) = 0$, for all $r \geq 0$. Since $\Psi^*(t,r) = \Psi(T-t,r)$ on $(0,T]$, we can then by virtue of Theorem 1.1 state:

Corollary 1.2 *Suppose R_t is a homogeneous Markov process given by (1.19). Then the function $\Phi^*(t,r) = 1 - \Psi^*(t,r)$ satisfies the partial integro-differential equation*

$$\begin{aligned}
&-\frac{\partial \Phi^*}{\partial t}(t,r) + \frac{\partial \Phi^*}{\partial r}(t,r)b(r) \\
&\qquad = \Phi^*(t,r)\lambda - \lambda \int_{\{z\,|\,r\ge z\}} \Phi^*(t,r-z)G(dz), \quad t,r>0.
\end{aligned} \tag{1.20}$$

Having evaluated Φ^* (numerically) over $(0,T]$, $T<\infty$, we can evaluate quantities such as
$E[\tau I(\tau<T) \,|\, R_0 = r]$ and $E[\tau \wedge T \,|\, R_0 = r]$, and higher moments. For instance

$$\begin{aligned}
E[\tau \wedge T \,|\, R_0 = r] &= \int_0^T \Phi^*(s,r)ds \\
&= \int_0^T \Phi(s,r)ds,
\end{aligned} \tag{1.21}$$

where the first equality sign follows by definition, and the second follows from $\Phi^*(t,r) = \Phi(T-t,r)$.

By defining $\Psi(r) = \lim_{t\uparrow\infty} \Psi^*(t,r)$, we can, by cancelling the differentiation w.r.t. t in (1.20), obtain an integro-differential equation for the non-ruin probability in infinite time (compare with the comments preceding (1.8)),

$$\Phi(r) = 1 - \Psi(r).$$

We can now state:

Corollary 1.3 *Suppose R_t is a homogeneous Markov process given by (1.19). Then $\Psi(R_{t\wedge\tau})$ is a (uniformly integrable) martingale, and the function $\Phi(r) = 1 - \Psi(r)$ satisfies the integro-differential equation*

$$\frac{d\Phi}{dr}(r)b(r) = \lambda\Phi(r) - \lambda\int_{\{z\,|\,r\geq z\}} \Phi(r-z)G(dz), \quad r > 0. \tag{1.22}$$

Asmussen and Petersen (1988) have established an integral equation for evaluating the probability of ruin in infinite time, when R_t is given by (1.19) with $b(r) > 0$, and $Z_i > 0$. Equation (1.22) seems not to appear in Asmussen and Petersen (1988) or Petersen (1989), but can easily be derived using their connection between the stationary density of the content dam process and the non-ruin probability. If, furthermore, $b(r)$ is independent of r, (1.22) leads to the well-known equation for the classical model, see e.g. Grandell (1990, p. 4).

There is one case where we can shift to a homogeneous Markov process when evaluating the probability of ruin in infinite time for a non-homogeneous Markov process:

Example 1.1. The classical model under discounting.
When considering ruin in infinite time, (1.14) can be reduced to an ordinary differential equation in r under suitable stationarity conditions: The risk process $X_t = \sum_{n=1}^{N_t} Z_n$ is a compound Poisson process with underlying intensity given by (1.18). Consider the economic environment where payments are discounted in accordance with a constant force of interest $\delta \neq 0$, so that the annual discount factor is $v = e^{-\delta}$. Premiums are paid continuously at a constant rate $b > 0$. The present value by time 0 of the surplus is given by

$$R_t = R_0 + b\int_0^t v^s ds - \sum_{n=1}^{N_t} v^{T_n} Z_n,$$

which is a time non-homogeneous $\mathcal{F}_t^N$-Markov process, and is of the form (1.1) with $b(t,r) \equiv bv^t,\ f(t,z) \equiv v^t z$. Due to the stationarity property of X_t and δ and since $T = \infty$, it should then be obvious that

$$\Psi(t,r) = \Psi(0, r/v^t), \tag{1.23}$$

which means that the probability of getting ruined after time t with reserve r is equivalent to getting ruined at time zero with initial reserve r/v^t. Therefore

$$\frac{\partial\Phi}{\partial t}(t,r) = r\delta v^{-t}\frac{\partial\Phi}{\partial r}(0, r/v^t). \tag{1.24}$$

Putting $t = 0$ in (1.14) and (1.24), we get (see also Delbaen and Haezendonck, 1987, pp. 105-107),

$$(b + \delta r)\frac{\partial \Phi}{\partial r}(0, r) = \lambda \Phi(0, r) - \lambda \int_0^r \Phi(0, r - z) G(dz).$$

Consequently, the ruin probability in an economic environment as described above can also be viewed as a ruin probability for a risk reserve consisting of the reserve dependent premium $b(r) = b + \delta r$ and risk process X_t. Obviously, this means that the ruin probability is reduced in the presence of positive interest.

This result holds over any time horizon $[0, T)$ also for a time dependent force of interest and for a general X_t process, but then (1.23) is no longer valid. This can be seen as follows: Let $\delta(t)$ be the time dependent force of interest assumed to be piecewise continuous, and let $\tilde{R}_t$ be the risk reserve consisting of the reserve dependent premium $b(t, r) = b(t) + \delta(t) r$, such that $\tilde{R}_t$ fulfills the differential equation

$$d\tilde{R}_t = b(t)dt + \delta(t)\tilde{R}_t dt - dX_t,$$

which has the solution

$$\tilde{R}_t = e^{\int_0^t \delta(s)ds} \left\{ \tilde{R}_0 + \int_0^t b(s) e^{-\int_0^s \delta(u)du} ds - \int_0^t e^{-\int_0^s \delta(u)du} dX_s \right\}.$$

The first time $\tilde{R}_t$ becomes negative over $[0, T)$ is equivalent to the first time the process in parentheses becomes negative, which is the risk reserve in the economic environment with interest rate $\delta(t)$ and premium rate $b(t)$. $\square$

5.1.2 Relations to other results

Below we shall discuss the results in the light of the set-up in Dassios and Embrechts (1989). The discussion is only applicable for $T < \infty$. They focus on a process Y_t, consisting of two components (η_t, Q_t), where η_t is set to 1 and $Q_t = R_t$, whenever $R_t > 0$, if ruin has not occurred by time t, otherwise $\eta_t = 0$ and Q_t is defined to be absorbed in 0. By finding h in the domain of the extended generator $\mathcal{A}$ for the Y_t process satisfying $\mathcal{A}h(t, y) = 0$, they obtain that $h(t, Y_t)$ becomes a martingale. Using an approach similar to that in Dassios and Embrechts (1989, p. 187), we conclude that $h(t, Y_t) = h(t, 1, Q_t) I(Q_t > 0)$ is a martingale, where $h(t, 1, r)$ satisfies an equation similar to (1.14) for all $t, r > 0$, and $h(t, 0, 0) = 0$. If, furthermore $T < \infty$ and h is chosen such as $h(T, 1, r) = 1$ for all $r > 0$, then by conditioning on $(\tau < T)$ and $(\tau \geq T)$, respectively, and using the martingale property of $h(t, Y_t)$, they obtain that

$$P(\tau \geq T \,|\, R_0 = r) = h(0, 1, r), \ \forall r > 0,$$

where we define $P(\tau \geq T \,|\, R_0 = 0) = \lim_{r \downarrow 0} h(0, 1, r)$. By virtue of Theorem 1.1 we allow to make the identification

$$h(t, 1, R_t) I(R_t \geq 0) = \Phi(t, R_t),$$

which not seems to have been dicussed in their paper.

5.1.3 Numerical illustrations

In this subsection we shall give some examples of numerical evaluation of (1.17). We discretize the integral term by use of a Simpson formula and solve the respective system of differential equations recursively over $[0, T)$ by use of the classical Runge-Kutta method.

We consider a risk reserve of the form

$$R_t = R_0 + c\,t - \sum_{i=1}^{N_t} Z_i, \tag{1.25}$$

where $R_0 \geq 0$ is the initial reserve, $c > 0$ is the constant premium rate, and we assume that the intensity is given by (1.18), where G is assumed to possess a continuous density g on $\mathcal{R}_+$. Then (1.17) reads for $t \in (0, T)$

$$\frac{d\tilde{\Phi}_u}{dt}(t) = \tilde{\Phi}_u(t)\lambda - \lambda \int_{\{z \mid u+c\,t \geq z\}} \tilde{\Phi}_{u-z}(t)\, g(z) dz. \tag{1.26}$$

For a fixed $U > 0$, we want to evaluate $\tilde{\Phi}_u(0)$ for some $u \in [0, U]$. Let $0 = t_0 < t_1 < \cdots < t_m = T$, be a partition of the interval $[0, T]$ with an equidistant division norm $h = t_i - t_{i-1}$. Starting with the initial condition $\tilde{\Phi}_u(T) = 1$, $u \geq -c\,T$, we evaluate recursively $\tilde{\Phi}_u(t_i)$ for $u \in [-c\,t_i, U]$, $i = m-1, m-2, \ldots, 0$, where the intervals $[-c\,t_i, U]$ also are divided into subintervals with division norm h.

For an example, assume that G is an exponential distribution function with mean 1, implying that

$$g(x) = e^{-x}, \;\; x \geq 0. \tag{1.27}$$

Table 1: Values of $\tilde{\Psi}_r(0) = 1 - \tilde{\Phi}_r(0)$ under (1.26), (1.18), (1.27) with parameters

$$U = 15, \;\; T = 1$$

r	$\lambda = 10$, $c = 11$, $h = 10$	$\lambda = 10$, $c = 11$, $h = 20$	$\lambda = 20$, $c = 22$, $h = 20$
0	0.794830	0.790128	0.835602
1	0.620574	0.616550	0.693935
2	0.475711	0.472393	0.570218
3	0.358370	0.355709	0.463685
4	0.265552	0.263470	0.373199
5	0.193722	0.192128	0.297356
6	0.139244	0.138048	0.234598
7	0.098692	0.097811	0.183309
8	0.069027	0.068390	0.141891
9	0.047674	0.047220	0.108829
10	0.032534	0.032217	0.082729
11	0.021951	0.021733	0.062344
12	0.014651	0.014503	0.046587
13	0.009678	0.009580	0.034528
14	0.006330	0.006266	0.025386
15	0.004101	0.004060	0.018521

The figures in Table 1 were evaluated by running a simple Pascal program on a Personal Computer. The choices of the parameters correspond to a safety loading on the premium of

$$\rho = 1 - \frac{c}{\lambda E[Z_1]} = 0.1.$$

Unfortunately, when comparing columns 1 and 2, there seems to be an inaccuracy, especially for smaller values of r.

5.2 More complex models

In this section we will discuss some models of relevance to insurance where the risk reserve is not necessarily Markov, but where a Markovization is feasible. We will study cases where the intensity is allowed to be a stochastic process.

Firstly, we will extend the model assumptions from Section 1 by allowing the intensity to depend on the history only via N_{t-}. This is indicated by writing $\lambda_t(N_{t-}, A)$. An example could be the following: Assume N_t is a homogeneous Poisson process with intensity λ and independent of $Z_1, Z_2, \dots$, which also are assumed mutually independent such that Z_n is assumed to be distributed in accordance with G_n. The intensity would then become

$$\lambda_t(N_{t-}, dz) = \lambda\, G_{N_{t-}+1}(dz).$$

We will consider the more general version of (1.1):

$$R_t = R_0 + \int_0^t b(s, R_s, N_s)ds - \int_0^t \int_{\mathcal{Z}} f(s, z)dN_s(dz), \tag{2.1}$$

where for each n, $(t,r) \to b(t,r,n)$ takes the role of b in (1.1). We could also allow f to depend on N_{t-}, but for simplicity we avoid this. Then (R_t, N_t) becomes a Markov process, and the martingale in (1.7) reads

$$M_{t\wedge\tau} = \Psi(t\wedge\tau, R_{t\wedge\tau}, N_{t\wedge\tau}), \tag{2.2}$$

where $\Psi(t,r,n) = P(\inf_{t\le s<T} R_s < 0 \,|\, R_t = r, N_t = n)$. The function $\Phi(t,r,n) = 1 - \Psi(t,r,n)$ will only change in t and r between the jumps and is assumed to be governed by the continuous partial derivatives, $\frac{\partial\Phi}{\partial t}(t,r,n)$ and $\frac{\partial\Phi}{\partial r}(t,r,n)$, respectively. Repeating the techniques and arguments used in Section 1, we obtain the following partial integro-differential equation for $\Phi(t,r,n)$:

$$\begin{aligned}&\frac{\partial\Phi}{\partial t}(t,r,n) + \frac{\partial\Phi}{\partial r}(t,r,n)b(t,r,n)\\ &\quad = \Phi(t,r,n)\lambda_t(n) - \int_{\{z\,|\,r\ge f(t,z)\}} \Phi(t, r-f(t,z), n+1)\lambda_t(n,dz).\end{aligned} \tag{2.3}$$

Using arguments similar to those following (1.15), (2.3) implies that for a non-negative function U_t satisfying the differential equation

$$\frac{dU_t}{dt} = b(t, U_t, n), \;\; n \ge 0,$$

$\Phi(t, U_t, n)$ satisfies the differential equation

$$\begin{aligned}d\Phi(t,U_t,n) &= \Phi(t,U_t,n)\lambda_t(n)dt\\ &\quad - \int_{\{z\,|\,U_t\ge f(t,z)\}} \Phi(t, U_t - f(t,z), n+1)\lambda_t(n,dz)dt,\end{aligned} \tag{2.4}$$

which leads to the following system of integro-differential equations in n:

$$\begin{aligned}\frac{d\tilde\Phi_u}{dt}(t,n) &= \tilde\Phi_u(t,n)\lambda_t(n)\\ &\quad - \int_{\{z\,|\,U_t\ge f(t,z)\}} \tilde\Phi_{u-f(t,z)}(t,n+1)\lambda_t(n,dz),\end{aligned} \tag{2.5}$$

where $\tilde\Phi_u(t,n) \equiv \Phi(t, u + \int_0^t b(s,U_s,n)ds, n)$, and $U_t = u + \int_0^t b(s,U_s,n)ds$. The initial condition becomes $\tilde\Phi_u(T,n) = 1$, $u \ge -\int_0^T b(s,U_s,n)ds$ and $n \ge 0$, and system (2.5) must finally be solved numerically. This problem shall not be pursued here.

Finally, we will study the probability of ruin in a Markovian environment. The set-up here is more general than that treated in Reinhard (1984) and Asmussen

(1989) since the processes involved can be of non-homogeneous Markov type and, further, the premium is allowed to depend on the environment and reserve.

Assume there is given an (observable) Markov jump process $(\Theta_t)_{t\geq 0}$, which for simplicity is assumed to have finite state space $\mathcal{J} = \{1, \ldots, J\}$ and satisfying $\Theta_0 = 1$. The intensity is assumed to fluctuate according to Θ_t, that is, it is assumed to be a function of Θ_t. To study this model in the framework of a point process, we let the marks represent either the pair of states (θ_{i-1}, θ_i), caused by a transition of Θ_t at time T_i, or represent the pair (Y_i, θ_i), where Y_i is some random variable assumed for simplicity to be non-negative and typically representing a claim amount. To compare the results here with those in Reinhard (1984), we will assume that these two kinds of event cannot coincide. We write $N(t, A)$ instead of $N_t(A)$ and decompose it into the two associated counting processes

$$N_i(t, B) = \sum_{k\geq 1} I(T_k \leq t,\ Y_k \in B,\ \Theta_{T_k} = i),\ B \in \mathcal{B}_+, \tag{2.6}$$

$$\tag{2.7}$$

$$N_{ij}(t) = \sum_{k\geq 1} I(T_k \leq t,\ \Theta_{T_k} = j,\ \Theta_{T_k-} = i),\ i \neq j. \tag{2.8}$$

$$\tag{2.9}$$

The natural filtration $\mathcal{F}_t^N$ can now be considered as generated by (2.6) and (2.8). As mentioned above, we assume that (2.6) and (2.8) cannot have common jumps.

The intensities of the counting processes (2.6), (2.8) are denoted $\lambda_i(t, B)$ and $\lambda_{ij}(t)$, respectively, and are assumed to depend on the history only via Θ_t.

Consider the risk reserve

$$R_t = R_0 + \int_0^t b_{\Theta_s}(s, R_s)ds - \sum_{i\in\mathcal{J}} \int_0^t \int_{\mathcal{R}_+} f_i(s, y)dN_i(s, dy),$$

where $b_i(t, r)$ takes the role of $b(t, r)$ in (1.1). Then (R_t, Θ_t) becomes a non-homogeneous Markov process, which implies that the martingale in (1.7) reads

$$M_{t\wedge\tau} = \Psi_{\Theta_{t\wedge\tau}}(t \wedge \tau, R_{t\wedge\tau}),$$

where $\Psi_i(t, r) = P(\inf_{t\leq s<T} R_s < 0 \,|\, R_t = r,\ \Theta_t = i)$.

Using the change of variable formula for $\Phi_{\Theta_t}(t, R_t) = 1 - \Psi_{\Theta_t}(t, R_t)$, we obtain

$$
\begin{aligned}
&\Phi_{\Theta_{t\wedge\tau}}(t\wedge\tau, R_{t\wedge\tau}) - \Phi_{\Theta_0}(0, R_0) \\
&= \int_0^{t\wedge\tau} \frac{\partial \Phi_{\Theta_s}}{\partial t}(s, R_s)ds + \int_0^{t\wedge\tau} \frac{\partial \Phi_{\Theta_s}}{\partial r}(s, R_s)b_{\Theta_s}(s, R_s)ds \\
&\quad + \sum_i \int_0^{t\wedge\tau} \int_{\mathcal{R}_+} [\Phi_i(s, R_{s-} - f_i(s,y)) - \Phi_i(s, R_{s-})]dN_i(s, dy) \\
&\quad + \sum_{i\neq j} \int_0^{t\wedge\tau} [\Phi_i(s, R_s) - \Phi_j(s, R_s)]dN_{ij}(s), \qquad (2.10)
\end{aligned}
$$

where $\sum_{i\neq j} = \sum_i \sum_{j\neq i}$. It is used in (2.10) that the counting processes do not have common jumps, and in particular we have replaced R_{t-} with R_t in the last term. Put $\bar{\lambda}_i(t) = \sum_{j\neq i} \lambda_{ij}(t)$ and let $\lambda_i(t) = \lambda_i(t, \mathcal{R}_+)$. Repeating the arguments leading to Theorem 1.1 we can obtain the following system of partial differential equations:

$$
\begin{aligned}
&\frac{\partial \Phi_i}{\partial t}(t, r) + \frac{\partial \Phi_i}{\partial r}(t, r)b_i(t, r) \\
&= \Phi_i(t, r)\lambda_i(t) - \int_{\{y \,|\, r \geq f_i(t,y)\}} \Phi_i(t, r - f_i(t, y))\lambda_i(t, dy) \\
&\quad + \Phi_i(t, r)\bar{\lambda}_i(t) - \sum_{j\neq i} \Phi_j(t, r)\lambda_{ij}(t) \\
&= \Phi_i(t, r)(\lambda_i(t) + \bar{\lambda}_i(t)) - \int_{\{y \,|\, r \geq f_i(t,y)\}} \Phi_i(t, r - f_i(t, y))\lambda_i(t, dy) \\
&\quad - \sum_{j\neq i} \Phi_j(t, r)\lambda_{ij}(t), \quad t \in (0, T), \quad r > 0. \qquad (2.11)
\end{aligned}
$$

In particular one sees that (2.11) reduces to an equation similar to (1.14) if Θ_t can admit only a single value.

An example of $\lambda_i(t, B)$ could be

$$\lambda_i(t, dy) = \lambda_i(t)F_i(dy),$$

where $\lambda_i(t)$ can be interpreted as a claims intensity for $N_i(t) \equiv N_i(t, \mathcal{R}_+)$, and F_i as a claim amount distribution depending on the state of Θ_t, where $N_i(t)$ and the claim amounts are conditionally independent given $\Theta_t = i$. Reinhard (1984) studied this case with $f_i(t, y) \equiv y$, and $\lambda_i(t)$, $\lambda_{ij}(t)$ independent of t and $b_i(t, r)$ independent of t and r.

Chapter 6

Diffusion approximations

In this chapter we will see applications of the advanced theory of stochastic calculus for identifying limiting distributions for a risk business. The limiting distributions will be of diffusion type, see below. We will view the process governing the risk business as a semimartingale, which is a process consisting of a sum of a local martingale and a process with paths of bounded variation over finite intervals. Furthermore, the process of bounded variation will be assumed continuous.

Diffusions approximations have mainly been studied in the classical model, where the jumps are governed by a homogeneous Poisson process and the size of the jumps represent claim amounts assumed to be i.i.d. (independent and identically distributed) and independent of the Poisson process. For this model, Grandell (1977) studied the probability of ruin by obtaining a diffusion approximation to a Gaussian process. The approximation is obtained by simultaneously increasing the initial reserve and the period of time and further letting the safety loading decrease, all of the same order. Asmussen (1984) uses the same idea, but also proposes a refinement of the approximation.

Sheike (1992) considers sequences of risk processes by summation over i.i.d. processes, and uses the martingale limit theorem to obtain a diffusion approximation when the number of risk units increases.

In Section 1, we stress the conditions to obtain a diffusion approximation for a sequence of semimartingales. In Subsection 1.1, we study some applications when the point processes are of Poisson type. In Subsection 1.2, we generalize the first part of Subsection 1.1 to the case where the point processes are of mixed Poisson type.

The approximations are obtained by showing weak convergence of the sequence of processes governing the risk business to a continuous Ito process. The sequence of processes are considered as elements in the space $\mathcal{D}$ of right-continuous functions with left-hand limits. This space is endowed with the Skorohod topology, and we write $X^{(n)} \xrightarrow{W} X$ when $X^{(n)}$ converges weakly to X as $n \to \infty$. When $X^{(n)}$, X are real valued random variables we instead write $X^{(n)} \xrightarrow{d} X$. Rigorous exposition on this topic can e.g. be found in Liptser and Shiryaev (1989).

A diffusion process appears in its general form as

$$Y_t = Y_0 + \int_0^t b(s, Y_s)ds + \int_0^t \sigma(s, Y_s)dB_s, \tag{0.1}$$

where the coefficients $b : \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ and $\sigma : \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ are Borel-measurable mappings, and $(B_t)_{t\geq 0}$ is a Brownian motion assumed to be given in advance on $(\Omega, \mathcal{F}, P)$. With suitable conditions on b and σ, see e.g. Øksendal (1992, pp. 48-49), and for a given random variable U, $E[U^2] < \infty$, independent of $\mathcal{F}_\infty^B = \sigma(B_s,\ s < \infty)$, there exists a (strong) solution to (0.1) with $Y_0 = U$, which is adapted to the filtration $\mathcal{F}_t^B = \sigma(B_s,\ s \leq t)$. For the cases to be studied, we will always assume that there exsists a unique solution to (0.1), such that the process I defined by

$$I_t = \int_0^t \sigma(s, Y_s)dB_s \tag{0.2}$$

becomes a well defined zero mean martingale, satisfying the Ito isometry (Øksendal 1992, pp. 18-21)

$$E\left[\int_0^t \sigma(s, Y_s)dB_s\right]^2 = E\left[\int_0^t \sigma^2(s, Y_s)ds\right], \quad t \in [0, T].$$

We will below only consider cases where σ is non-stochastic, implying that (0.2) becomes a Gaussian process (independent increments) with variance function

$$\phi(t) = \int_0^t \sigma^2(s)ds.$$

6.1 Conditions for weak convergence

Throughout we assume that each individual risk unit can be modelled as a semi-martingale

$$R_t = R_0 + A_t + M_t, \tag{1.1}$$

where M_t is the zero mean martingale

$$M_t = X_t^{(f)} - \int_0^t \int_{\mathcal{Z}} f(s, z)\lambda_s(dz)ds, \tag{1.2}$$

and A_t is an FV process with continuous paths, usually referred to as the drift term.

The bracket process of M_t (and also R_t) is assumed to exist forall $t < \infty$, and is given by

$$< M >_t = \int_0^t \int_{\mathcal{Z}} f^2(s,z)\lambda_s(dz)ds. \tag{1.3}$$

This process plays a crucial part in obtaining the diffusion approximations. In particular, (1.3) implies that M_t becomes a locally square integrable martingale.

Let now $N_t^{(n)}(A)$ be a sequence of marked point processes possessing $\mathcal{F}_t^{N^{(n)}}$-intensities $\lambda_t^{(n)}(A)$, where $\mathcal{F}_t^{N^{(n)}}$ denotes the filtration generated by $N_t^{(n)}(A)$. Furthermore, we consider a sequence $R^{(n)}$ of semimartingales

$$R_t^{(n)} = R_0^{(n)} + A_t^{(n)} + M_t^{(n)}, \tag{1.4}$$

where for each n, $A^{(n)}$ is as described in (1.1), and $M_t^{(n)}$ is a zero mean $\mathcal{F}_t^{N^{(n)}}$-martingale defined as in (1.2) with $\lambda_t(A)$ replaced by $\lambda_t^{(n)}(A)$.

We will stress the conditions to obtain a diffusion approximation for

$$Q^{(n)} = a_n R^{(n)},$$

where a_n is a sequence of positive constants tending to zero as n tends to infinity:

Assume there exists a function $L : \mathcal{R}_+ \to \mathcal{R}_+$ satisfying

$$\int_0^t L(s)ds < \infty, \quad t \in \mathcal{R}_+, \tag{1.5}$$

such that b in (0.1) satisfies the (linear growth) condition (Liptser and Shiryayev 1989, p. 623)

$$|b(t, Y_t)| \leq L(t)(1 + \sup_{s\leq t} |Y_s|), \quad t \in \mathcal{R}_+. \tag{1.6}$$

Assume also that (Liptser and Shiryayev 1989, pp. 624, 639)

$$\sup_{s\leq t} |a_n A_s^{(n)} - \int_0^s b(u, Q_u^{(n)})du| \xrightarrow{P} 0, \quad n \to \infty, \quad t \in \mathcal{R}_+, \tag{1.7}$$

where P denotes convergence in probability. Furthermore, the bracket process of $Q_t^{(n)}$ must converge (uniformly over finite intervals) in probability to the variance (process) of the limiting process:

$$\begin{aligned} &< Q^{(n)} >_t \\ &= a_n^2 \int_0^t \int_{\mathcal{Z}} f^2(s,z)\lambda_s^{(n)}(dz)ds \xrightarrow{P} \int_0^t \sigma^2(s)ds, \quad n \to \infty. \end{aligned} \tag{1.8}$$

Finally we need the Lindeberg condition:

$$\sum_{s\le t}(\triangle Q_s^{(n)})^2 I(|\triangle Q_s^{(n)}|>\epsilon)$$

$$= \quad a_n^2\int_0^t\int_{\mathcal{Z}} f^2(s,z)I(a_n|f(s,z)|>\epsilon)dN_s(dz) \xrightarrow{P} 0, \;\; \epsilon>0, \;\; n\to\infty. \tag{1.9}$$

Since convergence in mean implies convergence in probability, (1.9) is obtained whenever

$$a_n^2 E\left[\int_0^t\int_{\mathcal{Z}} f^2(s,z)I(a_n|f(s,z)|>\epsilon)\lambda_s^{(n)}(dz)ds\right] \longrightarrow 0, \;\; \epsilon>0, \;\; n\to\infty. \tag{1.10}$$

The Lindeberg condition ensures that the limiting process becomes continuous.

According to Liptser and Shiryayev (1989, pp. 624, 625, 639), we can then state:

Theorem 1.1 *Consider the sequence of processes given by (1.4). Assume that*

$$Q_0^{(n)} \xrightarrow{d} Y_0, \;\; n\to\infty.$$

Then under (1.5)-(1.9)

$$Q^{(n)} \xrightarrow{W} Y, \;\; n\to\infty,$$

where Y is the diffusion

$$Y_t = Y_0 + \int_0^t b(s,Y_s)ds - \int_0^t \sigma(s)dB_s.$$

6.1.1 The Poisson case

Throughout this section, we assume that the sequence of point processes are Poisson processes, and firstly we pay attention to the case where the intensities are given by

$$\lambda_t^{(n)}(dz) = \lambda^{(n)}G_t(dz), \tag{1.11}$$

such that $\lambda^{(n)}\to\infty$ as $n\to\infty$.

We then have:

Theorem 1.2 *Consider the sequence of processes given by (1.4), and choose* $a_n = (\lambda^{(n)})^{-1/2}$. *Assume that*

$$Q_0^{(n)} \xrightarrow{d} Y_0, \ \ n \to \infty.$$

Then under (1.5)-(1.7) and (1.11)

$$Q^{(n)} \xrightarrow{W} Y, \ \ n \to \infty,$$

where Y *is the diffusion*

$$Y_t = Y_0 + \int_0^t b(s, Y_s)ds - \int_0^t \sigma(s)dB_s,$$

with

$$\sigma^2(t) = \int_{\mathcal{Z}} f^2(t,z)G_t(dz).$$

Proof: It is readily checked that

$$< Q^{(n)} >_t = \int_0^t \int_{\mathcal{Z}} f^2(s,z)G_s(dz)ds,$$

which becomes deterministic and independent of n, and consequently (1.8) is fulfilled. Condition (1.10) follows by dominated convergence. □

As an example we can obtain:

Corollary 1.3 *Consider the sequence of processes given by*

$$\begin{aligned} R_t^{(n)} &= R_0^{(n)} + \int_0^t \delta_s R_s^{(n)} ds \\ &\quad + (1+\alpha^{(n)})\lambda^{(n)} \int_0^t \int_{\mathcal{Z}} f(s,z)G_s(dz)ds - \int_0^t \int_{\mathcal{Z}} f(s,z)N_s^{(n)}(dz), \end{aligned}$$

where $t \to \delta_t$ *from* $\mathcal{R}_+ \to \mathcal{R}$ *is assumed to be piecewise continuous. Let* $a_n = (\lambda^{(n)})^{-1/2}$, *and assume that*

$$Q_0^{(n)} \xrightarrow{d} Y_0, \ \ \alpha^{(n)}(\lambda^{(n)})^{1/2} \to 1, \ \ n \to \infty.$$

Then under (1.11)

$$Q^{(n)} \xrightarrow{W} Y, \ \ n \to \infty,$$

where Y *is the diffusion*

$$Y_t = Y_0 + \int_0^t (\mu(s) + \delta_s Y_s)ds - \int_0^t \sigma(s)dB_s,$$

with

$$\mu(t) = \int_{\mathcal{Z}} f(t,z)G_t(dz), \ \ \sigma^2(t) = \int_{\mathcal{Z}} f^2(t,z)G_t(dz).$$

Proof: Obviously $R^{(n)}$ becomes a sequence of semimartingales with drift term

$$A_t^{(n)} = \int_0^t \delta_s R_s^{(n)} ds + \alpha^{(n)} \lambda^{(n)} \int_0^t \int_{\mathcal{Z}} f(s,z) G_s(dz) ds.$$

Hence

$$(\lambda^{(n)})^{-1/2} A_t^{(n)} = \int_0^t \delta_s Q_s^{(n)} ds + \alpha^{(n)} (\lambda^{(n)})^{1/2} \int_0^t \int_{\mathcal{Z}} f(s,z) G_s(dz) ds,$$

so (1.7) is trivially fulfilled with

$$b(t,y) = \delta_t\, y + \int_{\mathcal{Z}} f(t,z) G_t(dz).$$

Condition (1.6) is readily checked since:

$$\begin{aligned} |b(t,Y_t)| &= |\delta_t\, Y_t + \int_{\mathcal{Z}} f(t,z) G_t(dz)| \\ &\le |\delta_t| \sup_{s \le t} |Y_s| + \int_{\mathcal{Z}} |f(t,z)| G_t(dz) \\ &\le \{\, |\delta_t| + \int_{\mathcal{Z}} |f(t,z)| G_t(dz) \,\}\, (1 + \sup_{s \le t} |Y_s|), \end{aligned}$$

so (1.5) and (1.6) are obtained with

$$L(t) = |\delta_t| + \int_{\mathcal{Z}} |f(t,z)| G_t(dz).$$

Condition (1.8)-(1.9) follow as discussed in Theorem 1.2. $\square$

Consequently, we can obtain:

Corollary 1.4 *Consider the sequence of zero mean $\mathcal{F}_t^{N^{(n)}}$-martingales given by*

$$M_t^{(n)} = \int_0^t \int_{\mathcal{Z}} f(s,z) dN_s^{(n)}(dz) - \lambda^{(n)} \int_0^t \int_{\mathcal{Z}} f(s,z) G_s(dz) ds.$$

Then

$$(\lambda^{(n)})^{-1/2} M^{(n)} \xrightarrow{W} Y, \;\; n \to \infty,$$

where Y is the Gaussian process

$$Y_t = \int_0^t \sigma(s) dB_s,$$

with

$$\sigma^2(t) = \int_{\mathcal{Z}} f^2(t,z) G_t(dz).$$

Proof: Conditions (1.5)-(1.7) are no longer needed, and (1.8)-(1.9) follow as above with $Q^{(n)} = (\lambda^{(n)})^{-1/2} M^{(n)}$. $\square$

We will now turn to a more complicated situation, namely where the intensities are given by

$$\lambda_t^{(n)}(dz) = \lambda^{(n)} G^{(n)}(dz), \tag{1.12}$$

which corresponds to the general homogeneous case for $N_t^{(n)}(A)$. We still assume that $\lambda^{(n)} \to \infty$ as $n \to \infty$.

To repeat the results above, we could face trouble in verifying the Lindeberg condition, since there is now a possibility that large jumps can occur as n increases.

To make the theory applicable, we furthermore restrict attention to treat only jump processes for the individual risk units of the form

$$X_t^{(f)} = \int_0^t \int_{\mathcal{R}} z f(s) dN_s(dz), \tag{1.13}$$

where $f : \mathcal{R}_+ \to \mathcal{R}$ is some Borel measurable mapping.

The case of interest is when

$$\liminf_n E[Z_n^2] > 0, \tag{1.14}$$

where

$$E[Z_n^2] = \int_{\mathcal{R}} z^2 G^{(n)}(dz),$$

is assumed to exist for all n.

We assume that there exist non-negative functions H, $\tilde{H}$ of finite variation, such that

$$\begin{aligned} &1 - G^{(n)}(z) \le H(z), \;\; z \ge 0, \;\; \int_{(0,\infty)} zH(z)dz < \infty, \\ &G^{(n)}(z) \le \tilde{H}(z), \;\; z < 0, \;\; -\int_{(-\infty,0)} z\tilde{H}(z)dz < \infty. \end{aligned} \tag{1.15}$$

We need the following Lemma:

Lemma 1.5 *Under (1.15) we have*

$$\begin{aligned} &x^2 H(x) \to 0, \;\; x \to \infty, \\ &x^2 \tilde{H}(x) \to 0, \;\; x \to -\infty. \end{aligned} \tag{1.16}$$

Proof: We only prove the result for $x \to \infty$.

It should be obvious that

$$H(x) \to 0, \;\; x \to \infty.$$

Switching the order of integration, we can then obtain

$$\int_0^\infty z^2 H(dz) = -2\int_0^\infty zH(z)dz.$$

Using integration by parts, we also get

$$x^2 H(x) = \int_0^x z^2 H(dz) + 2\int_0^x zH(z)dz.$$

Combining (1.17) and (1.17), the result follows by dominated convergence. □

We will find the following relation useful:

$$\begin{aligned} &\int_{\mathcal{R}} z^2 I(|z| > x) G^{(n)}(dz) \\ &= \;\; 2\int_{\mathcal{R}} zI(z > x)(1 - G^{(n)}(z))dz - 2\int_{\mathcal{R}} zI(z < -x)G^{(n)}(z)dz \\ &\quad + x^2[1 - G^{(n)}(x) + G^{(n)}(-x)], \;\; x \geq 0, \end{aligned} \tag{1.17}$$

which is obtained by switching the order of integration.

We can now state:

Theorem 1.6 *Consider the sequence of processes given by (1.4), where the jump part is replaced by (1.13) such that f is non-vanishing and bounded over finite intervals. Choose $a_n = (\lambda^{(n)} E[Z_n^2])^{-1/2}$, and assume that*

$$Q_0^{(n)} \xrightarrow{d} Y_0, \;\; n \to \infty.$$

Then under (1.5)-(1.7), (1.12) and (1.14)-(1.15)

$$Q^{(n)} \xrightarrow{W} Y, \;\; n \to \infty,$$

where Y is the diffusion

$$Y_t = Y_0 + \int_0^t b(s, Y_s)ds - \int_0^t f(s)dB_s.$$

Proof: Under (1.14) we have $a_n \to 0$ as $n \to \infty$. Obviously

$$< Q^{(n)} >_t = \int_0^t f^2(s)ds,$$

so (1.8) is trivially fulfilled.

Using first (1.17) and then (1.15), we can verify (1.10), since

$$\begin{aligned}
& a_n^2 \int_0^t \int_{\mathcal{R}} z^2 f^2(s) I(a_n |z f(s)| > \epsilon) \lambda^{(n)} G^{(n)}(dz)ds \\
= \quad & (E[Z_n^2])^{-1} \left[2 \int_0^t f^2(s) \int_{\mathcal{R}} z I(z > |f^{-1}(s)|\epsilon/a_n)(1 - G^{(n)}(z))dzds \right. \\
& -2 \int_0^t f^2(s) \int_{\mathcal{R}} z I(z < -|f^{-1}(s)|\epsilon/a_n) G^{(n)}(z))dzds \\
& \left. +(\epsilon/a_n)^2 \int_0^t \{1 - G^{(n)}(|f^{-1}(s)|\epsilon/a_n) + G^{(n)}(-|f^{-1}(s)|\epsilon/a_n)\}ds \right] \\
\leq \quad & 2(\liminf_n E[Z_n^2])^{-1} \left[2 \int_0^t f^2(s) \int_{\mathcal{R}} z I(z > |f^{-1}(s)|\epsilon/a_n) H(z)dzds \right. \\
& -2 \int_0^t f^2(s) \int_{\mathcal{R}} z I(z < -|f^{-1}(s)|\epsilon/a_n) \tilde{H}(z)dzds \\
& \left. +(\epsilon/a_n)^2 \int_0^t \{H(|f^{-1}(s)|\epsilon/a_n) + \tilde{H}(-|f^{-1}(s)|\epsilon/a_n)\}ds \right].
\end{aligned}$$

Using (1.14) and dominated convergence, the first two terms converge to zero as $n \to \infty$, and the last term also by (1.16). □

As an example, we can similarly to Corollary 1.4 state:

Corollary 1.7 *Consider the sequence of zero mean $\mathcal{F}_t^{N^{(n)}}$-martingales given by*

$$M_t^{(n)} = \int_0^t \int_{\mathcal{R}} z f(s) dN_s^{(n)}(dz) - \lambda^{(n)} \int_0^t \int_{\mathcal{R}} z f(s) G^{(n)}(dz)ds,$$

where f is non-vanishing and bounded over finite intervals. Then under (1.14)-(1.15) we have

$$(\lambda^{(n)} E[Z_n^2])^{-1/2} M^{(n)} \xrightarrow{W} Y, \quad n \to \infty,$$

where Y is the Gaussian process

$$Y_t = \int_0^t f(s) dB_s.$$

Proof: Conditions (1.5)-(1.7) are no longer needed, and (1.8)-(1.9) follow as above. $\square$

Example 1.1. Suppose there are given m independent risk processes $X_i^{(n)}(t)$, $i = 1, \ldots, m$, such that each $X_i^{(n)}(t)$ is a compound Poisson process with claim intensity $\lambda_i^{(n)}$ and claim amount distribution F_i. The total risk process of interest becomes $X^{(n)}(t) = \sum_{i=1}^m X_i^{(n)}(t)$ with $\mathcal{F}_t^N$-intensity given by

$$\begin{aligned}\lambda_t^{(n)}(dz) &= \sum_{i=1}^m \lambda_i^{(n)} F_i(dz) \\ &= \lambda^{(n)} G^{(n)}(dz),\end{aligned}$$

where $\lambda^{(n)} = \sum_{i=1}^m \lambda_i^{(n)}$, and $G^{(n)} = \frac{1}{\lambda^{(n)}} \sum_{i=1}^m \lambda_i^{(n)} F_i$,

such that $\int_{\mathcal{R}} z^2 F_i(dz) < \infty$. Since the intensity process determines the process, the risk process $X^{(n)}(t)$ can also be considered as a compound Poisson process.

Condition (1.14) is valid because

$$E[Z_n^2] \geq \min_{1 \leq i \leq m} \left\{ \int_{\mathcal{R}} z^2 F_i(dz) \right\}, \ \forall n,$$

and condition (1.15) is obtained with

$$H(z) = \sum_{i=1}^m (1 - F_i(z)), \quad \tilde{H}(z) = \sum_{i=1}^m F_i(z).$$

Hence, we can obtain diffusion approximations when the total (claims) intensity $\lambda^{(n)}$ tends to infinity. $\square$

6.1.2 The mixed Poisson case

In this section, we will generalize the first part of Subsection 1.1 to a case where the intensity process of $N_t^{(n)}(A)$ is stochastic and depending only on the number of jumps $N_t^{(n)}$. Namely, assume there is given a sequence of non-negative random variables $\Theta^{(n)}$ with mean one, such that for given $\Theta^{(n)}$, $N_t^{(n)}(A)$ admits the intensity

$$\tilde{\lambda}_t^{(n)}(dz) = \lambda^{(n)} \Theta^{(n)} G_t(dz).$$

In particular, we see that for given $\Theta^{(n)}$, $N_t^{(n)}$ is a Poisson process with intensity $\lambda^{(n)}\Theta^{(n)}$, which is also referred to as the mixed Poisson process. The $\mathcal{F}_t^{N^{(n)}}$-intensity of $N_t^{(n)}(A)$ then reads

$$\lambda_t^{(n)}(dz) = \lambda^{(n)} \, E(\Theta^{(n)} \mid N_t^{(n)}) G_t(dz). \tag{1.18}$$

The technical part in the following is to show convergence of the bracket process in (1.8), which now reads

$$< Q^{(n)} >_t = a_n^2 \, \lambda^{(n)} \int_0^t \int_{\mathcal{Z}} f^2(s,z) E(\Theta^{(n)} \mid N_s^{(n)}) G_s(dz) ds. \tag{1.19}$$

A tool to showing convergence of (1.19) is characteristic function, which already has been introduced in Chapter 4, Example 2.4. We need a more general formulation:

Consider a sequence of $\mathcal{F}_t^{N^{(n)}}$-measurable semimartingales $Y_t^{(n)}$, assumed to possess an intensity measure $\lambda_t^{(n)}(A)$. Let $\mathcal{C}$ denote the complex valued numbers. Any complex number y is written $y = Re\, y + i\, Im\, y$, where $Re\, y$ and $Im\, y$ denote the real and imaginary part of y, respectively. Also, keep in mind that

$$e^{ix} = \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} = \cos x + i \sin x, \ \forall x \in \mathcal{R}.$$

According to Jacod and Shiryaev (1987, p. 75 and p. 415-428), one must look for a sequence of complex valued processes $\psi_t^{(n)}(u), \ u \in \mathcal{R}$, of the form

$$\psi_t^{(n)}(u) = iuB_t^{(n)} - \frac{u^2}{2} C_t^{(n)} + \int_0^t \int_{\mathcal{R}} (e^{iuy} - 1 - iuy) \lambda^{(n)}(dy) ds,$$

where $B_t^{(n)}$ and $C_t^{(n)}$ are cadlag processes such that

$$M_t^{(n)} = e^{iuY_t^{(n)}} / e^{\psi_t^{(n)}(u)},$$

is an $\mathcal{F}_t^{N^{(n)}}$-complex valued martingale for each n. The triplet $(B^{(n)}, C^{(n)}, \lambda^{(n)}(A))$ is for each n called the characteristics of $Y^{(n)}$, and

$$\phi_t^{(n)}(u) = e^{\psi_t^{(n)}(u)}$$

is called the characteristic function.

The sequence $(Y_t^{(n)})_{t \geq 0}$ then converges in distribution to a Gaussian process $Y = (Y_t)_{t \geq 0}$ with triplet $(\mu_t, \sigma_t^2, 0)$ if

$$e^{\psi_t^{(n)}(u)} \xrightarrow{P} \phi_t(u), \ \ n \to \infty,$$

uniformly in u and t over finite intervals, where

$$\phi_t(u) = e^{iu\mu_t - \frac{u^2}{2} \sigma_t^2}$$

is the characteristic function of the normal distribution with mean μ_t and variance σ_t^2.

Lemma 1.8 *Let $h : \mathcal{R}_+ \times \mathcal{Z} \to \mathcal{R}$ be some Borel measurable mapping fulfilling*

$$\int_0^t \int_{\mathcal{Z}} |h(s,z)| G_s(dz) ds < \infty, \quad t > 0.$$

Then

$$(\lambda^{(n)})^{-1} \int_0^t \int_{\mathcal{Z}} h(s,z)(dN_s^{(n)}(dz) - \lambda^{(n)} E(\Theta^{(n)} \mid N_s^{(n)}) G_s(dz) ds) \xrightarrow{P} 0, \tag{1.20}$$

$$(\lambda^{(n)})^{-1} \int_0^t \int_{\mathcal{Z}} h(s,z)(dN_s^{(n)}(dz) - \lambda^{(n)} \Theta^{(n)} G_s(dz) ds) \xrightarrow{P} 0, \tag{1.21}$$

$$\int_0^t \int_{\mathcal{Z}} h(s,z) E(\Theta^{(n)} \mid N_s^{(n)}) G_s(dz) ds - \Theta^{(n)} \int_0^t \int_{\mathcal{Z}} h(s,z) G_s(dz) ds \xrightarrow{P} 0, \tag{1.22}$$

uniformly in t over finite intervals as $\lambda^{(n)} \to \infty$ for $n \to \infty$.

Proof: Only (1.20) is proved, because (1.21) is proved similarly by conditioning on $\Theta^{(n)}$, and (1.22) follows by subtraction of (1.20) from (1.21).

By Example 2.4 of Chapter 4, the sequence $\phi_t^{(n)}(u)$ of characteristic functions of the sequence in (1.20) is given by choosing

$$\begin{aligned}
\psi_t^{(n)}(u) \quad &= \quad \lambda^{(n)} \int_0^t \int_{\mathcal{Z}} (e^{iu(\lambda^{(n)})^{-1} h(s,z)} - 1) E(\Theta^{(n)} \mid N_s^{(n)}) G_s(dz) ds \\
&\quad - iu \int_0^t \int_{\mathcal{Z}} h(s,z) E(\Theta^{(n)} \mid N_s^{(n)}) G_s(dz) ds.
\end{aligned} \tag{1.23}$$

Using a series expansion for $x \to e^{ix}$ in (1.23),

$$\psi_t^{(n)}(u) = \int_0^t \int_{\mathcal{Z}} r_s^{(n)}(u,z) E(\Theta^{(n)} \mid N_s^{(n)}) G_s(dz) ds,$$

where

$$\begin{aligned}
r_t^{(n)}(u,z) \quad &= \quad \lambda^{(n)} \sum_{k=2}^{\infty} \frac{[iu(\lambda^{(n)})^{-1} h(t,z)]^k}{k!} \\
&= \quad \sum_{k=1}^{\infty} \frac{[iu\, h(t,z)]^{k+1}}{(k+1)!} (\lambda^{(n)})^{-k}.
\end{aligned}$$

For fixed t, z, the mapping $u \to r_t^{(n)}(u,z)$ from $\mathcal{R}$ to $\mathcal{C}$ is differentiable with derivative

$$\begin{aligned}
r_t^{(n)}{}'(u,z) &= ih(t,z)\sum_{k=1}^{\infty}\frac{[iu\,h(t,z)]^k}{k!}(\lambda^{(n)})^{-k} \\
&= ih(t,z)(e^{iu(\lambda^{(n)})^{-1}h(t,z)}-1).
\end{aligned}$$

Also, $r_t^{(n)}(0,z)=0$. By the mean value theorem there exist ξ, $\eta \in (0,u)$ (may depend on n) such that

$$\begin{aligned}
r_t^{(n)}(u,z) &= [Re\, r_t^{(n)}{}'(\xi,z) + Im\, r_t^{(n)}{}'(\eta,z)]u \\
&= iu\,h(t,z)[cos(\xi(\lambda^{(n)})^{-1}h(t,z)) + isin(\eta(\lambda^{(n)})^{-1}h(t,z)) - 1].
\end{aligned}$$

Thus

$$|r_t^{(n)}(u,z)| \le 3|u||h(t,z)|.$$

It is clear that $r_t^{(n)}(u,z) \to 0$ as $n \to \infty$, and dominated convergence then implies that

$$\int_0^t \int_{\mathcal{Z}} |r_s(u,z)|G_s(dz)ds \to 0, \quad n \to \infty.$$

As convergence in mean implies convergence in probability, we get

$$\int_0^t \int_{\mathcal{Z}} |r_s^{(n)}(u,z)|E(\Theta^{(n)}\,|\,N_s^{(n)})G_s(dz)ds \xrightarrow{P} 0, \quad n \to \infty,$$

and since the integrand is non-negative this convergence is uniform in t and u over finite intervals. This shows that the characteristic function of the process in (1.20) converges uniformly in probability to $\phi_t(u) \equiv 1$, which is the characteristic function of a Gaussian process with drift and variance equal to zero. The limit process is thus equivalent to zero. By basic theory of weak convergence (see e.g. Liptser and Shiryayev, 1989 p. 506) it then also follows that

$$S_t^* = \sup_{s\in(0,t]}\{(\lambda^{(n)})^{-1}\int_0^s \int_{\mathcal{Z}} h(\tau,z)(dN_\tau^{(n)}(dz) - \lambda^{(n)}E(\Theta^{(n)}\,|\,N_\tau)G_\tau(dz)d\tau)\},$$

converges in distribution to a Gaussian process with no drift and variance, which completes the proof of (1.20). □

We can make use of the lemma by further assuming

$$\Theta^{(n)} \xrightarrow{P} 1, \quad n \to \infty. \tag{1.24}$$

We can now state theorems similar to those in the Poisson case:

Theorem 1.9 *Consider the sequence of processes given by (1.4), and choose* $a_n = (\lambda^{(n)})^{-1/2}$*. Assume that*

$$Q_0^{(n)} \xrightarrow{d} Y_0, \ \ n \to \infty.$$

Then under (1.5)-(1.7), (1.18) and (1.24)

$$Q^{(n)} \xrightarrow{W} Y, \ \ n \to \infty,$$

where Y *is the diffusion*

$$Y_t = Y_0 + \int_0^t b(s, Y_s)ds - \int_0^t \sigma(s)dB_s,$$

with

$$\sigma^2(t) = \int_{\mathcal{Z}} f^2(t,z)G_t(dz).$$

Proof: Using (1.24) and (1.22) with $h = f^2$, we immediately get

$$\begin{aligned} < Q^{(n)} >_t &= \int_0^t \int_{\mathcal{Z}} f^2(s,z)E(\Theta^{(n)} \mid N_s^{(n)})G_s(dz)ds \\ &\xrightarrow{P} \int_0^t \int_{\mathcal{Z}} f^2(s,z)G_s(dz)ds, \ \ n \to \infty, \end{aligned}$$

which proves (1.8). Finally, as in Theorem 1.2, condition (1.10) is fulfilled by dominated convergence. □

Furthermore, we can state:

Corollary 1.10 *Consider the sequence of processes given by*

$$\begin{aligned} R_t^{(n)} &= R_0^{(n)} + \int_0^t \delta_s R_s^{(n)} ds \\ &\quad + (1+\alpha^{(n)})\lambda^{(n)} \int_0^t \int_{\mathcal{Z}} f(s,z)E(\Theta^{(n)} \mid N_s^{(n)})G_s(dz)ds \\ &\quad - \int_0^t \int_{\mathcal{Z}} f(s,z)N_s^{(n)}(dz), \end{aligned}$$

where $t \to \delta_t$ *from* $\mathcal{R}_+ \to \mathcal{R}$ *is assumed piecewise continuous. Let* $a_n = (\lambda^{(n)})^{-1/2}$*, and assume that*

$$Q_0^{(n)} \xrightarrow{d} Y_0, \quad \alpha^{(n)}(\lambda^{(n)})^{1/2} \to 1, \ \ n \to \infty.$$

Then under (1.18) and (1.24)

$$Q^{(n)} \xrightarrow{W} Y, \quad n \to \infty,$$

where Y is the diffusion

$$Y_t = Y_0 + \int_0^t (\mu(s) + \delta_s Y_s) ds - \int_0^t \sigma(s) dB_s,$$

with

$$\mu(t) = \int_{\mathcal{Z}} f(t,z) G_t(dz), \quad \sigma^2(t) = \int_{\mathcal{Z}} f^2(t,z) G_t(dz).$$

Proof: As in the proof of Corollary 1.3, we write

$$A_t^{(n)} = \int_0^t \delta_s R_s^{(n)} ds + \alpha^{(n)} \lambda^{(n)} \int_0^t \int_{\mathcal{Z}} f(s,z) E(\Theta^{(n)} \mid N_s^{(n)}) G_s(dz) ds.$$

Hence

$$\begin{aligned}(\lambda^{(n)})^{-1/2} A_t^{(n)} &= \int_0^t \delta_s Q_s^{(n)} ds \\ &\quad + \alpha^{(n)} (\lambda^{(n)})^{1/2} \int_0^t \int_{\mathcal{Z}} f(s,z) E(\Theta^{(n)} \mid N_s^{(n)}) G_s(dz) ds,\end{aligned}$$

and using (1.22), (1.7) is trivially satisfied with

$$b(t,y) = \delta_t\, y + \int_{\mathcal{Z}} f(t,z) G_t(dz).$$

Condition (1.6) follows as in the proof of Corollary 1.3, and (1.8)-(1.9) as above. □

Finally, we can obtain:

Corollary 1.11 *Consider the sequence of zero mean $\mathcal{F}_t^{N^{(n)}}$-martingales given by*

$$M_t^{(n)} = \int_0^t \int_{\mathcal{Z}} f(s,z) dN_s^{(n)}(dz) - \lambda^{(n)} \int_0^t \int_{\mathcal{Z}} f(s,z) E(\Theta^{(n)} \mid N_s^{(n)}) G_s(dz) ds.$$

Then under (1.24)

$$(\lambda^{(n)})^{-1/2} M^{(n)} \xrightarrow{W} Y, \quad n \to \infty,$$

where Y is the Gaussian process

$$Y_t = \int_0^t \sigma(s) dB_s,$$

with

$$\sigma^2(t) = \int_{\mathcal{Z}} f^2(t,z) G_t(dz).$$

Proof: Conditions (1.5)-(1.7) are no longer needed, and (1.8)-(1.9) follow as above with $Q^{(n)} = (\lambda^{(n)})^{-1/2} M^{(n)}$. $\square$

Remark that we could alternatively prove Corollary 1.11 by use of characteristic function along the lines of the proof of Lemma 1.8.

Condition (1.24) states that if we can obtain the value of the sequence in the limit, we are, roughly speaking, back in Poisson case of Subsection 1.1.

Example 1.2. Consider n independent risk units, such that for given Ψ_i, unit i makes jumps (claims) with intensity $\lambda\Psi_i$, $\lambda > 0$. The latent random variables $\Psi_1, \Psi_2, \ldots, \Psi_n$ are assumed to be i.i.d. with $E[\Psi_1] = 1$.

Since the risk units behave independently, the $\mathcal{F}_t^{N^{(n)}}$-intensity process for the total risk process becomes

$$\lambda_t^{(n)}(dz) = \lambda\, E\left(\sum_{i=1}^{n} \Psi_i \mid N_t^{(n)}\right) G_t(dz),$$

where $N_t^{(n)}$ represents the total number of jumps. This can obviously be written on the form (1.18) with

$$\Theta^{(n)} = \frac{1}{n}\sum_{i=1}^{n} \Psi_i, \quad \lambda^{(n)} = n\lambda,$$

where $E[\Theta^{(n)}] = 1$, and for given $\sum_{i=1}^{n} \Psi_i$, $N_t^{(n)}$ is a sequence of Poisson processes with intensity

$$\lambda \sum_{i=1}^{n} \Psi_i = \lambda^{(n)}\Theta^{(n)}.$$

The law of large numbers ensures that

$$\Theta^{(n)} \xrightarrow{P} E[\Psi_1] = 1, \quad n \to \infty,$$

and (1.24) is then fulfilled. $\square$

Acknowledgements

This thesis was mainly carried out at the Laboratory of Actuarial Mathematics, Univ. of Copenhagen during the period 1991-1994. The author would like to thank the supervisor, Professor Ragnar Norberg, for helpful comments and encouragement during the whole process.

Between 1992 and 1993 the author visited, for a period of 8 months, the department of Electrical Engineering, Imperial College, London. In this connection the author would like to thank M.H.A. Davis for an inspiring stay.

Also the author is grateful to Martin Jacobsen, Statistical Department, Univ. of Copenhagen, for showing interest in parts of the work and for his always helpful attitude and comments.

Any errors and omissions are, of course, the responsibility of the author.

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