Point Processes, Martingales and Exit Times in Risk Theory

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Abstract

In this paper, we shall propose some martingale results and study their applications in risk theory. A celebrated topic is the problem of evaluating the probability of ruin, which can be considered as a special case of evaluating distributions of first exit (entry) times from (to) Borel sets. First, we shall mention how the time of ruin can be related to the Doléans equation, implying that we can view the non-ruin probability as a mean of an exponential. For infinite time ruin in the classical case, the problem partly reduces to integrating the tail of the distribution function for the individual claim amounts. The risk business shall be modelled as a stochastic process consisting of a continuous and discrete (jump) part. The building stones are delivered by the theory of marked point processes and associated martingale theory, hereunder the important concept of an intensity measure.

Furthermore, we shall propose some ideas on establishing integro-differential equation for evaluating the distribution of the involved jump part (which typically represents the total amount of claims) for some fixed period of time.

Also, we shall mention a martingale result valid for fairly general (Markov) processes, that can lead to (integro) differential equations for evaluating the distribution of exit times.

Keywords: Marked point process, Martingale, Doléans equation, Integro-differential equation, Ruin probability, Exit time, Boundary value problems.

1 Introduction

A martingale approach for studying ruin probabilities (the classical Lunberg inequality) dates back to Gerber (1973). A more refined formulation is given by Dassios and Embrechts (1989) who use results by Davis (1984) for the extended generator for PD (piecewise deterministic) Markov processes to obtain funtionals that become martingales; see also Møller (1992). Asmussen and Nielsen (1994) proposed a generalization of the Lundberg inequality to the case where the premium depends on the reserve.

This paper proposes some (new) martingale results and techniques, appropriate for analysing insurance models, initiated by the author's PhD-thesis, see Møller (1991), (1992), (1993) and (1995). Martingales can be useful for evaluating means of important functionals of processes related to the insurance business. However this requires that the relevant martingales exist and are observed.

In Section 2, we outline some elements of point process theory which is used in the sequel. In Section 3 we propose a (new) martingale result for studying the probability of ruin. We first observe that the indicator for the time of ruin can be related to the Doléans equation. It is assumed that ruin only can occur at jumps of the involved risk business. Explicit expressions seem not obtainable, but for the classical model (Subsection 3.3), we can for the ruin probability in infinite time obtain a more tractable expression. However, this requires uniform integrability of the martingale. In Section 4, we shall see that the martingale approach also leads to a way of finding an integro-differential equation for evaluating the distribution of the jump part when it has independent increments. Further results and techniques on integro-differential equations for evaluating the distribution of the jump part are discussed. Finally, we shall in Section 5 propose an important martingale result, which can be useful for identifying the distribution of an exit time (for fairly general Markov processes) from a differential equation akin to the form of the extended generator for the involved Markov process.

2 Some elements of point process theory

Assume throughout that there is given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, where the space $(\Omega, \mathcal{F}, P)$ is assumed to be complete, and the family $(\mathcal{F}_t)_{t\geq 0}$ of σ-algbras satisfies the usual conditions.

Let $\mathcal{R}^n$ denote the n-dimensional euclidian space, equipped with the usual Borel σ-algebra $\mathcal{B}^n$. For $n = 1$, we write $\mathcal{R}$ and $\mathcal{B}$. Further, we let $\mathcal{R}_+$ denote the non-negative half line (also equipped with the Borel σ-algebra). For the definitions and the martingale properties below, we refer to Brémaud (1981, pp. 8, 9, 234-236).

A marked point process is a sequence of stochastic pairs $(T_n, Z_n)_{n\geq 1}$, where $T_1, T_2, \ldots$ denote the non-negative points representing times of occurrence of some phenomena $Z_1, Z_2, \ldots$ called the marks, which are assumed to take values in some space $\mathcal{Z}$ endowed with a σ-algebra $\mathcal{E}$. In non-life insurance the points could typically represent times of occurrence of claims which could take the role of the marks.

Associated with the sequence of pairs $(T_n, Z_n)_{n\geq 1}$ we define the cadlág (right continuous with left-hand limit) process

$$N(t, A) = \sum_{n=1}^{\infty} I(T_n \leq t,\ Z_n \in A), \tag{2.1}$$

which counts the number of jumps in the time interval $[0, t]$ with values in $A \in \mathcal{E}$, where $I(A)$ denotes the indicator of a set $A \in \mathcal{F}$. We abbreviate $N(t) = N(t, \mathcal{Z})$. The natural filtration is defined as

$$\mathcal{F}_t^N = \sigma(N(s, A),\ s \leq t,\ A \in \mathcal{E}).$$

Assume that $N(t, A)$ possesses an $\mathcal{F}_t^N$-adapted (each $N(t, A)$ is $\mathcal{F}_t^N$-measurable) cadlág intensity process $\lambda_t(A)$, assumed to be bounded over finite intervals, informally defined by

$$\lambda_t(A)dt = P(N(dt, A) = 1 \,|\, \mathcal{F}_{t-}^N) + o(dt), \tag{2.2}$$

where $\mathcal{F}_{t-}^N = \vee_{s<t}\mathcal{F}_s^N$ is the information prior to time t and $o(h)/h \to 0$ for $h \to 0$. We abbreviate $\lambda_t = \lambda_t(\mathcal{Z})$ for the intensity of the $N(t)$ process. It can be more informative to write the intensity on the form

$$\lambda_t(A) = \lambda_t \int_A G_t(dz), \tag{2.3}$$

where G_t is a probability, $\int_{\mathcal{Z}} G_t(dz) = 1$, and is interpreted as the conditional probability given all information prior to time t and that a jump occurred at time t, that the associated mark will belong to $[z, z + dz]$. In the sequel we shall write $\int_a^b$, $\int_{\mathcal{Z}}$ for $\int_{(a,b]}$ and $\int_{z\in\mathcal{Z}}$, respectively.

Assume $\mathcal{F}_t^N \subset \mathcal{F}_t$ for all $t \geq 0$. Then the process

$$M_t = \int_0^t \int_{\mathcal{Z}} H(s, z)(N(ds, dz) - \lambda_s(dz)ds),$$

where H is an $\mathcal{F}_t$-predictable (indexed by $\mathcal{Z}$) process, becomes a zero mean $\mathcal{F}_t$-martingale whenever

$$E\left[\int_0^t \int_{\mathcal{Z}} |H(s, z)|\lambda_s(dz)ds\right] < \infty.$$

In particular

$$E\left[\int_0^t \int_{\mathcal{Z}} H(s, z)N(ds, dz)\right] = E\left[\int_0^t \int_{\mathcal{Z}} H(s, z)\lambda_s(dz)ds\right].$$

In the sequel it should be sufficient to know that, in particular, any process with left-continuous or deterministic paths (indexed by $\mathcal{Z}$) is predictable.

3 The non-ruin probability as an exponential

3.1 The probability of ruin and the Doléans equation

Consider a marked point process $(T_n, Y_n)_{n\geq 1}$, where the Y_n are assumed to be real valued. Assume in this section that $\mathcal{F}_t^N \subset \mathcal{F}_t$ for all $t \geq 0$.

Consider the risk business

$$R_t = R_0 + C_t - \sum_{i=1}^{N(t)} Y_i, \quad Y_i \in \mathcal{R}, \tag{3.1}$$

where C_t is an $\mathcal{F}_t$-adapted process, and R_0 is $\mathcal{F}_0$-measurable. Furthermore, we assume that C_t is continuous, satisfying $C_0 = 0$. The C_t process should be interpreted as a premium income process whereas $X_t \equiv \sum_{i=1}^{N(t)} Y_i$ could represent the total amount of claims occurred over $[0, t]$ with $Y_1, Y_2, \ldots$ representing the individual claim amounts. Then Y_n would normally take only non-negative values. Decomposing X_t in accordance with the jump times and the size of the jumps, we can write

$$X_t = \int_0^t \int_{\mathcal{R}} y\, N(ds, dy).$$

The time of ruin in its classical sense is defined as

$$\tau = \inf\{t \geq 0 \,|\, R_t < 0\},$$

and becomes an $\mathcal{F}_t$-stopping time, that is, $\{\tau \leq t\} \in \mathcal{F}_t$, $\forall t \geq 0$. Assume throughout that C_t only take non-negative values, which means that ruin only occurs at the jumps of R_t. We can then write

$$\begin{aligned} I(\tau \leq t) &= I(\tau = 0) + \sum_{0<s\leq t} I(\tau \geq s,\, R_s < 0) \\ &= I(\tau = 0) + \int_0^t \int_{\mathcal{R}} I(\tau \geq s,\, R_{s-} < y)\, N(ds, dy), \end{aligned} \tag{3.2}$$

where we in the last equality sign have decomposed after the size of the jumps. Define the cadlág processes

$$Z_t = I(\tau > t), \;\; Q_t = \int_0^t \int_{\mathcal{R}} I(R_{s-} < y)\, N(ds, dy),$$

and then rewrite (3.2) as

$$Z_t = I(\tau > 0) - \int_0^t Z_{s-}\, dQ_s,$$

which is an example of a Doléans equation, see e.g. Liptser and Shiryayev (1989, p. 122). Since Q_t is purely discrete, the solution is given as

$$\begin{aligned} Z_t &= I(\tau > 0) \prod_{0<s\leq t} [1 - \Delta Q_s] \\ &= I(\tau > 0) \prod_{0<s\leq t} \left[1 - \int_{\mathcal{R}} I(R_{s-} < y)\, N(ds, dy)\right], \end{aligned}$$

where $\Delta Q_t = Q_t - Q_{t-}$. We can now state:

Theorem 3.1 *The process*

$$M_t = I(\tau > t) \exp\left(\int_0^t \lambda_s \overline{G}_s(R_s) ds\right), \tag{3.3}$$

is an $\mathcal{F}_t$-martingale whenever

$$E\left[\exp\left(\int_0^t \lambda_s \overline{G}_s(R_s)ds\right)\right] < \infty, \tag{3.4}$$

where $\overline{G}_t(x) = \int_{\mathcal{R}} I(y > x)G_t(dy)$.

Proof: Since M_t is equivalent to writing

$$M_t = I(\tau > t) \exp\left(\int_0^t \int_{\mathcal{R}} I(R_s < y)\lambda_s(dy)ds\right),$$

we obtain by use of integration by parts and (3.2), that

$$\begin{aligned} M_t &= M_0 + \int_0^t M_s \int_{\mathcal{R}} I(R_s < y)\lambda_s(dy)ds - \int_0^t M_{s-} \int_{\mathcal{R}} I(R_{s-} < y)N(ds, dy) \\ &= M_0 - \int_0^t \int_{\mathcal{R}} M_{s-}\, I(R_{s-} < y)(N(ds, dy) - \lambda_s(dy)ds), \end{aligned}$$

which is a stochastic integral of an $\mathcal{F}_t$-predictable process w.r.t. to the martingale measure $U(dt, dy) = N(dt, dy) - \lambda_t(dy)dt$, which implies that M_t becomes an $\mathcal{F}_t$-martingale. $\square$

Throughout we shall write $P_x(A)$, $x \in \mathcal{R}$, for the conditional probability measure $P(A \,|\, R_0 = x)$, and we write E_x for the expectation w.r.t. P_x.

Consequently, we obtain:

Theorem 3.2 *The non-ruin probability satisfies the relation*

$$P_x(\tau > t)\, E_x\left[\exp\left(\int_0^t \lambda_s \overline{G}_s(R_s)ds\right) \,\middle|\, \tau > t\right] = 1, \quad t, x \geq 0.$$

Proof: Using the martingale property of M_t together with the fact that $M_0 = I(\tau > 0) = 1$ for $R_0 \geq 0$, we get that $E_x[M_t] = 1$ for $x \geq 0$, and the assertion follows by conditioning upon $\{\tau > t\}$ and $\{\tau \leq t\}$ in (3.3). $\square$

Due to the martingale property of M_t, we can for any $T < \infty$ and $x \geq 0$, introduce a new probability measure on $(\Omega, \mathcal{F})$ by

$$\tilde{P}_x(A) = E_x[M_T\, I(A)]. \tag{3.5}$$

We can then view the non-ruin probability in the light of:

Corollary 3.3 *For any $t \in [0, T]$, the non-ruin probability satisfies*

$$P_x(\tau > t) = \tilde{E}_x\left[\exp\left(-\int_0^t \lambda_s \overline{G}_s(R_s)ds\right)\right], \quad x \geq 0, \tag{3.6}$$

where $\tilde{E}_x$ is the expectation w.r.t. $\tilde{P}_x$.

Proof: Using the martingale property, we obtain

$$\begin{aligned} \tilde{E}_x\left[\exp\left(-\int_0^t \lambda_s \overline{G}_s(R_s)ds\right)\right] &= E_x\left[M_T \exp\left(-\int_0^t \lambda_s \overline{G}_s(R_s)ds\right)\right] \\ &= E_x\left[E_x\{M_T \,|\, \mathcal{F}_t\} \exp\left(-\int_0^t \lambda_s \overline{G}_s(R_s)ds\right)\right] \\ &= E_x\left[M_t \exp\left(-\int_0^t \lambda_s \overline{G}_s(R_s)ds\right)\right] \\ &= P_x(\tau > t), \quad x \geq 0, \end{aligned}$$

which proves the assertion. $\square$

Note that (3.6) is consistent in the sense that if $\hat{T} \neq T$, say $\hat{T} > T$, and we let $\hat{P}_x$ be the measure in (3.5) corresponding to $\hat{T}$, and $\hat{E}_x$ the respective mean, then

$$\tilde{E}_x\left[\exp\left(-\int_0^t \lambda_s \overline{G}_s(R_s)ds\right)\right] = \hat{E}_x\left[\exp\left(-\int_0^t \lambda_s \overline{G}_s(R_s)ds\right)\right], \quad t \in [0,T].$$

Repeating the steps in the proof above, we also see that

$$\tilde{P}_x(\tau > t) = 1, \quad t \in [0,T], \; x \geq 0.$$

The mean in (3.6) seems not possible to evaluate, but the relation can perhaps be applicable for simulation studies.

The martingale in (3.3) is P_x-uniformly integrable if and only if $M_t \to M_\infty$ in $L^1(\Omega, \mathcal{F}_t, P_x)$ (the space of P_x-integrable random variables) as $t \to \infty$, see e.g. Liptser and Shiryayev (1989, p. 20), where

$$M_\infty = I(\tau = \infty)\exp\left(\int_0^\infty \lambda_s \overline{G}_s(R_s)ds\right).$$

In particular, we then obtain that $E_x[M_\infty] = 1$, $x \geq 0$, and hence we could put $T = \infty$ in the definition of $\tilde{P}_x$. Also, uniform integrability implies that $P_x(\tau = \infty) > 0$, since otherwise we cannot have that $E_x[M_\infty] = 1$, for $x \geq 0$. Since

$$\sup_{t \geq 0} M_t = \exp\left(\int_0^\tau \lambda_s \overline{G}_s(R_s)ds\right),$$

a sufficient and natural condition for uniform integrability is then

$$E_x\left[\exp\left(\int_0^\tau \lambda_s \overline{G}_s(R_s)ds\right)\right] < \infty.$$

Under uniform integrability, we can repeat the arguments in Theorem 3.2 and Corollary 3.3 to state:

Theorem 3.4 *Suppose M_t is a uniformly integrable martingale. Then, the non-ruin probability in infinite time satisfies the relation*

$$P_x(\tau = \infty)\, E_x\left[\exp\left(\int_0^\infty \lambda_s \overline{G}_s(R_s)ds\right)\middle|\, \tau = \infty\right] = 1, \quad x \geq 0,$$

or with $T = \infty$ in (3.5)

$$P_x(\tau = \infty) = \tilde{E}_x\left[\exp\left(-\int_0^\infty \lambda_s \overline{G}_s(R_s)ds\right)\right], \quad x \geq 0.$$

3.2 The intensity process for the time of ruin

Alternatively, we can view the ruin problem in the framework of a simpel survival model such that 'alive' corresponds to non-ruin and 'dead' to ruin.

Differentiation in (3.6) gives for $t \in (0,T)$:

$$\begin{aligned}
\frac{d}{dt}P_x(\tau > t) &= -\tilde{E}_x\left[\lambda_t \overline{G}_t(R_t)\exp\left(-\int_0^t \lambda_s \overline{G}_s(R_s)ds\right)\right] \\
&= -E_x[\,I(\tau > t)\,\lambda_t \overline{G}_t(R_t)\,] \\
&= -E_x[\,\lambda_t \overline{G}_t(R_t)\,|\,\tau > t\,]P_x(\tau > t),
\end{aligned} \tag{3.7}$$

and since $T < \infty$ was arbitrary, we see that these relations hold for all $t \geq 0$, and could also be obtained directly by taking the mean in (3.2). Solving this differential equation with the initial condition $P_x(\tau > 0) = 1$ for $x \geq 0$, we get that

$$P_x(\tau > t) = \exp\left(- \int_0^t E_x[\, \lambda_s \overline{G}_s(R_s) \,|\, \tau \geq s \,] ds \right), \quad x \geq 0.$$

The process

$$A_t = I(\tau > t)\, \lambda_t \overline{G}_t(R_t),$$

is the $\mathcal{F}_t^N$-intensity process of the 1-dimensional counting process $\tilde{N}(t) = I(\tau \leq t)$, and the process

$$\tilde{A}_t = I(\tau > t)\, E_x[\, \lambda_t \overline{G}_t(R_t) \,|\, \tau \geq t \,],$$

is the $\tilde{\mathcal{F}}_t \equiv \sigma(\tilde{N}(s),\, s \leq t)$ intensity process of $\tilde{N}(t)$, obtained by $E_x[A_t \,|\, \tilde{\mathcal{F}}_t]$. So the non-ruin probability function $t \to P_x(\tau > t)$, $x \geq 0$, can be expressed as a 'survival function', with the deterministic 'mortality rate' $\mu_x(t)$, where

$$\mu_x(t) = E_x[\, \lambda_t \overline{G}_t(R_t) \,|\, \tau \geq t \,], \quad x \geq 0.$$

3.3 Infinity time ruin in the classical case

The classical model is when $(N(t))_{t \geq 0}$ is assumed to be a homogeneous Poisson process with constant intensity $\lambda > 0$, and $Y_1, Y_2, \ldots$ are non-negative i.i.d. (independent and identically distributed) random variables representing the individual claim amounts, and are assumed to be independent of $N(t)$. We denote their common distribution function by G, and assume that

$$E[Y_1] = \int_0^\infty \overline{G}(u) du < \infty.$$

The classical risk business is of the form

$$R_t = R_0 + c\, t - \sum_{i=1}^{N(t)} Y_i,$$

where $c > 0$ is the constant premium rate. The intensity measure of $N(t, A)$ now reads

$$\lambda_t(dy) = \lambda\, G(dy).$$

We shall assume that $c > \lambda E[Y_1]$, meaning that we are operating with a safety loading of

$$\rho = \frac{c}{\lambda E[Y_1]} - 1.$$

By the strong law of large numbers this implies that

$$\frac{R_t}{t} \to \rho \lambda E[Y_1] > 0, \quad as \;\; t \to \infty, \;\; a.s.$$

Then since $E[Y_1] < \infty$, we have

$$\int_0^\infty \overline{G}(R_s) ds < \infty, \;\; a.s.$$

Furthermore it is clear that (3.4) is satisfied and hence (3.3) becomes a martingale.

Introduce the function

$$H(x) = \int_0^x \overline{G}(u) du, \;\; x \geq 0,$$

with the understanding that $H(x) = 0$ for $x \le 0$. Then H is continuous on the whole axis. We will assume that H is differentiable with derivative $H'(x) = \overline{G}(x)$, $x > 0$, which is satisfied whenever $\overline{G}(x)$ is continuous. With the understanding $\overline{G}(x) = 0$ for $x \le 0$, H then becomes continuously differentiable on the whole axis with derivative

$$H'(x) = \overline{G}(x), \quad x \in \mathcal{R}.$$

Using the change of variable formula, we can then obtain

$$\begin{aligned} H(R_t) - H(R_0) &= c\int_0^t H'(R_s)ds + \sum_{0<s\le t} \{H(R_s) - H(R_{s-})\} \\ &= c\int_0^t \overline{G}(R_s)ds + \int_0^t \{H(R_s) - H(R_{s-})\}\, N(ds). \end{aligned}$$

Since $R_t \to \infty$ as $t \to \infty$, we get

$$\int_{R_0}^{\infty} \overline{G}(u)du = c\int_0^{\infty} \overline{G}(R_s)ds + \int_0^{\infty} \{H(R_s) - H(R_{s-})\}\, N(ds).$$

Thus

$$\lambda \int_0^{\infty} \overline{G}(R_s)ds = \frac{\lambda}{c}\int_{R_0}^{\infty} \overline{G}(u)du + \frac{\lambda}{c}\int_0^{\infty} \{H(R_{s-}) - H(R_s)\}\, N(ds),$$

and then

$$M_\infty = I(\tau = \infty)\exp\left(\frac{\lambda}{c}\int_{R_0}^{\infty} \overline{G}(u)du\right)\exp\left(\frac{\lambda}{c}\int_0^{\infty} \{H(R_{s-}) - H(R_s)\}\, N(ds)\right).$$

We obtain:

Theorem 3.5 *Suppose M_t is a unifomly integrable martingale. Then, the non-ruin probability in infinite time is given by*

$$P_x(\tau = \infty) = \rho_x \exp\left(-\frac{\lambda}{c}\int_x^{\infty} \overline{G}(u)du\right), \quad x \ge 0,$$

where

$$\rho_x^{-1} = E_x\left[\exp\left(\frac{\lambda}{c}\int_0^{\infty} \{H(R_{s-}) - H(R_s)\}\, N(ds)\right)\middle|\, \tau = \infty\right],$$

or with $T = \infty$ in (3.5)

$$P_x(\tau = \infty) = \tilde{\rho}_x \exp\left(-\frac{\lambda}{c}\int_x^{\infty} \overline{G}(u)du\right), \quad x \ge 0,$$

where

$$\tilde{\rho}_x = \tilde{E}_x\left[\exp\left(\frac{\lambda}{c}\int_0^{\infty} \{H(R_s) - H(R_{s-})\}\, N(ds)\right)\right].$$

Proof: Follows by the property $E_x[M_\infty] = 1$, $x \ge 0$. $\square$

If necessary, we can always write

$$\int_0^t \{H(R_s) - H(R_{s-})\}\, N(ds) = \int_0^t \int_{\mathcal{R}} \{H(R_{s-} - y) - H(R_{s-})\}\, N(ds, dy),$$

to obtain a predictable integrand.

4 The distribution of the jump process

First, we shall assume that $Y_n \geq 0$ for all n, implying that G_t is a probability on $\mathcal{R}_+$. Secondly, we assume that

$$C_t \equiv 0.$$

It is then obvious that

$$\begin{aligned} I(\tau \leq t) &= I(\inf\{s \leq t:\, R_0 - X_s < 0\}) \\ &= I(X_t > R_0), \end{aligned}$$

and hence the probability of ruin w.r.t. P_x in finite time reduces to finding the distribution of the jump process X_t. The martingale in (3.3) now modifies to

$$M_t^* = I(X_t \leq R_0) \exp\left(\int_0^t \lambda_s \overline{G}_s(R_0 - X_s)ds\right),$$

and in particular

$$E_x\left[I(X_t \leq x) \exp\left(\int_0^t \lambda_s \overline{G}_s(x - X_s)ds\right)\right] = 1, \quad x \geq 0.$$

As above we can for any $T < \infty$ introduce the measure

$$P_x^*(A) = E_x[\, M_T^* \, I(A)\,], \quad x \geq 0,$$

and we let E_x^* denote the mean w.r.t. P_x^*. As for (3.6), we obtain

$$P(X_t \leq x) = E_x^*\left[\exp\left(-\int_0^t \lambda_s \overline{G}_s(x - X_s)ds\right)\right], \quad t \in [0,T], \quad x \geq 0.$$

This is again a complicated mean value, and seems not of much value at first sight. However, we can at least obtain an integro-differential equation for $t \to P(X_t \leq x)$, when $N(t, A)$ are Poisson processes, implying that the intensity measure becomes deterministic, such that $G_t = 1 - \overline{G}_t$ now has the interpretation

$$G_t(x) = P(Y_n \leq x \,|\, T_n = t), \quad \forall n \geq 1.$$

Put $F(t,x) = P(X_t \leq x)$, and as for (3.7), we then obtain

$$\begin{aligned} \frac{dF}{dt}(t,x) &= -E_x^*\left[\lambda_t \overline{G}_t(x - X_t) \exp\left(-\int_0^t \lambda_s \overline{G}_s(x - X_s)ds\right)\right] \\ &= -\lambda_t\, F(t,x) + \lambda_t\, E_x^*\left[G_t(x - X_t) \exp\left(-\int_0^t \lambda_s \overline{G}_s(x - X_s)ds\right)\right] \\ &= -\lambda_t\, F(t,x) + \lambda_t\, E_x\left[\, I(X_t \leq x)\, G_t(x - X_t)\,\right] \\ &= -\lambda_t\, F(t,x) + \lambda_t \int_0^x G_t(x-y)\, F(t,dy) \\ &= -\lambda_t\, F(t,x) + \lambda_t \int_0^x F(t, x-y)\, G_t(dy), \quad t \in (0,T), \end{aligned} \tag{4.1}$$

and since T is arbitrary the integro-differential equation holds for all $t \geq 0$.

Since the cadlág process $I(X_t \leq x)$ is purely discrete, we could also establish (4.1) by first writing (the change of variable formula)

$$\begin{aligned} I(X_t \leq x) - I(x \geq 0) &= \sum_{0<s\leq t} \{I(X_s \leq x) - I(X_{s-} \leq x)\} \\ &= \int_0^t \int_{\mathcal{R}} \{I(X_{s-} + y \leq x) - I(\hat{X}_{s-} \leq x)\} N(ds, dy), \end{aligned} \tag{4.2}$$

where we have decomposed to obtain an $\mathcal{F}_t^N$-predictable integrand. Taking the mean on both sides and using that the intensity is deterministic, we arrive at (4.1). Note that such mathematical steps do not require that $Y_n \geq 0$, see below.

In cases where $N(t, A)$ do not possess independent increments, it seems that we instead must use a 'backward' approach, obtained by first introducing the cadlág process

$$\begin{aligned} \hat{X}_t &= \sum_{i=N(t)+1}^{N(T)} Y_i \\ &= \int_t^T \int_{\mathcal{R}} y \, N(ds, dy), \quad t \in [0, T], \end{aligned}$$

where we throughout use the convention $\sum_{i=N_t+1}^{N_t} Y_i = 0$ for any $t \geq 0$.

For the sake of illustration, we shall assume that the intensity process of $N(t, A)$ fluctuates in accordance with a Markovian environment. That is, we assume there is given an (observable) Markov jump process $(\Theta_t)_{t\geq 0}$, assumed to have finite state space $\mathcal{J} = \{1, \ldots, J\}$, such that $\lambda_t(A)$ can depend on the history at time t only via Θ_t. To study this model in the framework of a marked point process, we let the points T_i represent the jump times of either X_t or Θ_t, such that the corresponding marks are represented by the pairs (Y_i, θ_i) and (θ_{i-1}, θ_i), respectively, where $\Theta_{T_i} = \theta_i \in \mathcal{J}$. To avoid ambiguity, we shall assume that these two kind of events cannot occur simultaneously. Introduce the counting processes

$$N_i(t, B) = \sum_{k\geq 1} I(T_k \leq t, \ Y_k \in B, \ \Theta_{T_k} = i), \quad B \in \mathcal{B}, \quad i \in \mathcal{J}.$$

We assume that there for each $i \in \mathcal{J}$ exist a deterministic measure $\lambda_i(t, B)$ on $\mathcal{B}$, such that the intensity process of $N_i(t, B)$ is given by

$$P(N_i(dt, dy) = 1 \,|\, \mathcal{F}_{t-}^N) + o(dt) = \lambda_i(t, dy) I(\Theta_t = i) dt.$$

Furthermore, we define

$$P_{ij}(s, t) = P(\Theta_t = j \,|\, \Theta_s = i), \quad s \leq t,$$

the transition probabilities of $(\Theta_t)_{t\geq 0}$, and we assume that the transition intensities $\lambda_{ij}(t)$, $i \neq j$, exist, defined by

$$\lambda_{ij}(t) = \lim_{h \searrow 0} \frac{P_{ij}(t, t+h)}{h}.$$

Define the function

$$\hat{F}_i(t, x) = P(\hat{X}_t \leq x \,|\, \Theta_t = i), \quad i \in \mathcal{J}, \quad t \in [0, T].$$

Throughout we write $\sum_i$ and $\sum_{i\neq j}$ instead of $\sum_{i\in\mathcal{J}}$ and $\sum_{i\in\mathcal{J}} \sum_{j\neq i}$, respectively.

The process $I(\hat{X}_t \le r)$, $r \in \mathcal{R}$, is purely discrete and cadlág, satisfying $\hat{X}_T = 0$, so following the steps in (4.2), we obtain

$$\begin{aligned} I(r \ge 0) - I(\hat{X}_t \le r) &= \sum_{t<s\le T} \{I(\hat{X}_s \le r) - I(\hat{X}_{s-} \le r)\} \\ &= \sum_i \int_t^T \int_{\mathcal{R}} \{I(\hat{X}_s \le r) - I(\hat{X}_s + y \le r)\} N_i(ds, dy), \end{aligned}$$

where we have not decomposed to obtain an $\mathcal{F}_t^N$-predictable (left-continuous) integrand, so it seems not of much use to take the mean directly. However, using the Markov property together with the relation $E[E(\ \mid \mathcal{F}_s^N) \mid \mathcal{F}_t^N] = E(\ \mid \mathcal{F}_t^N)$ for $s \ge t$, we obtain by taking conditional mean, that

$$\begin{aligned} I(r \ge 0) - P_{\Theta_t}(\hat{X}_t \le r) &= E\left[\sum_i \int_t^T \int_{\mathcal{R}} \{I(\hat{X}_s \le r) - I(\hat{X}_s \le r - y)\} N_i(ds, dy) \middle| \mathcal{F}_t^N \right] \\ &= E\left[\sum_i \int_t^T \int_{\mathcal{R}} \{P_i(\hat{X}_s \le r) - P_i(\hat{X}_s \le r - y)\} N_i(ds, dy) \middle| \mathcal{F}_t^N \right] \\ &= E\left[\sum_i \int_t^T \int_{\mathcal{R}} \{\hat{F}_i(s, r) - \hat{F}_i(s, r - y)\} \lambda_i(s, dy) I(\Theta_s = i) ds \middle| \Theta_t \right], \end{aligned} \tag{4.3}$$

where the last equality sign follows by the martingale property of the process

$$O_t = \sum_i \int_t^T \int_{\mathcal{R}} \{\hat{F}_i(s, r) - \hat{F}_i(s, r - y)\} (N_i(ds, dy) - \lambda_i(s, dy) I(\Theta_s = i) ds).$$

Using the transition probabilities, we can modify (4.3) to

$$F_k(t, r) = I(r \ge 0) - \sum_i \int_t^T \int_{\mathcal{R}} \{\hat{F}_i(s, r) - \hat{F}_i(s, r - y)\} \lambda_i(s, dy) P_{ki}(t, s) ds. \tag{4.4}$$

Using then Kolmogorov's backward differential equations

$$\frac{dP_{ij}}{dt}(t, u) = \overline{\lambda}_i(t) P_{ij}(t, u) - \sum_{l \ne i} \lambda_{il}(t) P_{lj}(t, u),$$

where $\overline{\lambda}_i(t) = \sum_{j \ne i} \lambda_{ij}(t)$, we can state:

Theorem 4.1 *Over the continuity points of $\lambda_i(t, B)$ and $\lambda_{ij}(t)$, the functions $t \to \hat{F}_k(t, r)$ satisfy the system of differential equations*

$$\begin{aligned} \frac{d\hat{F}_i}{dt}(t, r) &= \hat{F}_i(t, r)(\lambda_i(t) + \overline{\lambda}_i(t)) - \int_{\mathcal{R}} \hat{F}_i(t, r - y) \lambda_i(t, dy) \\ &\quad - \sum_{j \ne i} \hat{F}_j(t, r) \lambda_{ij}(t), \quad t \in (0, T), \quad i \in \mathcal{J}, \quad r \in \mathcal{R}, \end{aligned}$$

where $\lambda_i(t) = \lambda_i(t, \mathcal{R})$.

Proof: Follows by differentiation in (4.4). $\square$

The integral equations read

$$\hat{F}_i(t,r) = I(r \geq 0)e^{-\int_t^T(\overline{\lambda}_i(s)+\lambda_i(s))ds} + \sum_{j\neq i}\int_t^T e^{-\int_t^\eta(\overline{\lambda}_i(s)+\lambda_i(s))ds}\hat{F}_j(\eta,r)\lambda_{ij}(\eta)d\eta$$

$$+\int_t^T\int_{\mathcal{R}} e^{-\int_t^\eta(\overline{\lambda}_i(s)+\lambda_i(s))ds}\hat{F}_i(\eta,r-y)\lambda_i(\eta,dy)d\eta.$$

Consequently, one would observe that the process

$$\hat{M}_t = \hat{F}_{\Theta_t}(t,x-X_t), \quad x\in\mathcal{R},$$

becomes an $\mathcal{F}_t^N$-martingale over $[0,T]$.

Note that the integrals

$$\int_{\mathcal{R}}\hat{F}_i(t,r-y)\lambda_i(t,dy),$$

above reduces to

$$\int_0^r\hat{F}_i(t,r-y)\lambda_i(t,dy),$$

in the case where the Y_n are assumed to take only non-negative values, which is convenient in a numerical implementation.

Finally, we shall present a martingale approach which seems convenient for analysing boundary value problems, hereunder the distribution of an exit time.

5 A martingale property related to exit times

Let $Y=(Y_t)_{t\geq 0}$ be an $\mathcal{F}_t$-adapted cadlág process, taking values in $\mathcal{R}^n$.

For a $D\in\mathcal{B}^n$, we define

$$\tau_D = \inf\{t\geq 0|\, Y_t\notin D\}, \quad D\in\mathcal{B}^n, \tag{5.1}$$

the first exit time from D, which is a Markov time, that is, $\{\tau_D\leq t\}\in\mathcal{F}_t$. We assume now that Y_t is an $\mathcal{F}_t$-Markov process, that is, $\sigma(Y_s,\, s\geq t)$ and $\mathcal{F}_t$ are independent given Y_t. We fix a period of time $T\leq\infty$ and define the conditional probabilities

$$\Psi(t,y) = P(\inf_{t\leq s<T}\{Y_s\notin D\}\,|\,Y_t=y), \;\; y\in\mathcal{R}^n.$$

By the Markov property, we have

$$\Psi(t,Y_t) = P(\inf_{t\leq s<T}\{Y_s\notin D\}\,|\,\mathcal{F}_t),$$

and furthermore, by definition,

$$\Psi(t,y) = 1, \;\; y\notin D, \;\; t\in[0,T). \tag{5.2}$$

Also, we have the initial condition

$$\Psi(T,y) = 0, \;\; y\in D, \;\; T<\infty.$$

For any $t \in [0,T)$, we write

$$\begin{aligned} I(\tau_D < T) &= I(\tau_D \le t) + I(t < \tau_D < T) \\ &= I(\tau_D \le t) + I(\tau_D > t) I(\inf_{t \le s < T} \{Y_s \notin D\}). \end{aligned} \tag{5.3}$$

Defining $M_t = P(\tau_D < T \,|\, \mathcal{F}_t) = E(I(\tau_D < T) \,|\, \mathcal{F}_t)$, and taking conditional expectation w.r.t. $\mathcal{F}_t$ in (5.3) and using the Markov property, we get

$$\begin{aligned} M_t &= I(\tau_D \le t) + P(t < \tau_D < T \,|\, \mathcal{F}_t) \\ &= I(\tau_D \le t) + I(\tau_D > t) P(\inf_{t \le s < T} \{Y_s \notin D\} \,|\, \mathcal{F}_t) \\ &= I(\tau_D \le t) + I(\tau_D > t) \Psi(t, Y_t). \end{aligned} \tag{5.4}$$

If Ψ is a continuous function from $\mathcal{R}_+ \times D$ to $[0,1]$, then M_t becomes cadlág with left-hand limits

$$M_{t-} = I(\tau_D < t) + I(\tau_D \ge t) \Psi(t, Y_{t-}).$$

In the sequel, we will always assume that Ψ is chosen such that M_t becomes cadlág.

Inserting $t \wedge \tau_D$ in (5.4), we obtain that

$$\begin{aligned} M_{t \wedge \tau_D} &= I(\tau_D \le t) + I(\tau_D > t) \Psi(t \wedge \tau_D, Y_{t \wedge \tau_D}) \\ &= \Psi(t \wedge \tau_D, Y_{t \wedge \tau_D}), \quad t \in [0,T], \end{aligned} \tag{5.5}$$

where the first equality sign follows by $I(\tau_D \le t \wedge \tau_D) = I(\tau_D \le t)$, and the second by (5.2): $\Psi(\tau_D, Y_{\tau_D}) = 1$, $\tau_D < T$, and in the case $\tau_D \ge T$, (5.5) is trivially satisfied. Thus we have obtained (optional sampling) that $\Psi(t \wedge \tau_D, Y_{t \wedge \tau_D})$ becomes a (bounded) $\mathcal{F}_t$-martingale over $[0,T]$.

Another relation: Using the optional sampling theorem, we obtain by taking conditional expectation on both sides in (5.5) w.r.t. the $\mathcal{F}_{t \wedge \tau_D}$-measurable stochastic variable $(t \wedge \tau_D, Y_{t \wedge \tau_D})$, that

$$P(\tau < T \,|\, t \wedge \tau_D, Y_{t \wedge \tau_D}) = \Psi(t \wedge \tau_D, Y_{t \wedge \tau_D}).$$

More generally: Fix an arbitrary $t' \in [0,T)$ and define

$$\tau'_D = \inf\{t \ge t' \,|\, Y_t \notin D\}, \tag{5.6}$$

which is the first exit time after t', and repeat the steps above to obtain that

$$M'_{t \wedge \tau'_D} = \Psi(t \wedge \tau'_D, Y_{t \wedge \tau'_D}), \quad t \in [t', T], \tag{5.7}$$

becomes an $\mathcal{F}_t$-martingale, where

$$\begin{aligned} M'_t &= P(t' < \tau'_D < T \,|\, \mathcal{F}_t) \\ &= I(\tau'_D \le t) + I(\tau'_D > t) \Psi(t, Y_t), \quad t \in [t', T]. \end{aligned} \tag{5.8}$$

We observe the relations (we only illustrate with τ_D and M_t)

$$M_{\tau_D} = I(\tau_D < T), \quad M_{\tau_D} = E(M_{\tau_D} \,|\, \mathcal{F}_{\tau_D -}),$$

since τ_D is $\mathcal{F}_{\tau_D -}$-measurable, and $\mathcal{F}_{\tau_D -} \subset \mathcal{F}_{\tau_D}$, see Brémaud (1981, p. 3). If furthermore τ_D is a predictable stopping time then (see Liptser and Shiryayev, 1989, p. 21)

$$M_{\tau_D -} = E(M_{\tau_D} \,|\, \mathcal{F}_{\tau_D -}),$$

and therefore

$$M_{\tau_D -} = M_{\tau_D}.$$

So in particular for $\tau_D < \infty$ and predictable, we have

$$\Psi(\tau_D, Y_{\tau_D}) = \Psi(\tau_D -, Y_{\tau_D -}),$$

meaning that we do not have a discontinuity of Ψ at the bondary of the domain D.

For an illustration consider the PD (piecewise-deterministic) Markov process

$$R_t = R_0 + \int_0^t b(s, R_s) ds - \sum_{i=1}^{N(t)} Y_i,$$

where the mapping $(t, r) \to b(t, r)$ from $\mathcal{R}_+ \times \mathcal{R}$ to $\mathcal{R}$, is assumed to be piecewise continuous in t and r, and could represent premium income or annuity payments to the insured. To obtain the Markov property we will assume that the intensity of $N(t, A)$ is deterministic ($N(t, A)$ is a Poisson process for each A).

The time of ruin

$$\tau = \inf\{t \geq 0 : R_t < 0\},$$

corresponds to τ_{D^*}, where $D^* = \{r \in \mathcal{R} : r \geq 0\}$. Also we define

$$\tau' = \inf\{t \geq t' : R_t < 0\},$$

the first time of exit from D^* after time t'. It is then possible to state:

Theorem 5.1 *For any fixed $t' \in [0, T)$ the process $\Psi(t \wedge \tau', R_{t \wedge \tau'})$ is a (uniformly integrable) martingale over $[t', T]$. Suppose that $(t, r) \to \Phi(t, r) = 1 - \Psi(t, r)$ from $(0, T) \times (0, \infty)$ to $[0, 1]$ has continuous partial derivatives, denoted $\frac{\partial \Phi}{\partial t}(t, r)$ and $\frac{\partial \Phi}{\partial r}(t, r)$, respectively. Then over the continuity points of $\lambda_t(A)$ and $b(t, r)$, $\Phi(t, r)$ satisfies the partial integro-differential equation*

$$\begin{aligned}
&\frac{\partial \Phi}{\partial t}(t, r) + \frac{\partial \Phi}{\partial r}(t, r) b(t, r) \\
&\qquad = \Phi(t, r)\lambda_t - \int_{\{y \mid r \geq y\}} \Phi(t, r - y)\lambda_t(dy), \quad t \in (0, T), \quad r > 0.
\end{aligned} \tag{5.9}$$

Proof: The first part follows by (5.7). The second part follows by first introducing the stopping time

$$\tau'_o = \inf\{t \geq t' : R_t \leq 0\},$$

which is the first time after t' that R_t exits the interior of D^*. Then use the change of variable technique, as described in Møller (1995), on the martingale

$$\tilde{M}_t = \Phi(t \wedge \tau' \wedge \tau'_o, R_{t \wedge \tau' \wedge \tau'_o}), \quad t \in [t', T].$$

The reason for introducing τ'_o, is that we assumed differentiability of Φ only on the interior of D^*. $\square$

Equation (5.9) implies that $\Phi(t, R_t)$ for $R_t > 0$ satisfies $\mathcal{A}\Phi = 0$, where $\mathcal{A}$ is the extended generator of R_t, but note that $\Phi(t, R_t)$ in general not becomes a martingale.

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