# A Point Process Approach to Inventory Models

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**Abstract**

The aim of the present paper is to make use of the modern theory of point processes to study optimal solutions for single-item inventory. Demand for goods is assumed to occur according to a compound Poisson process and production occurs continuously and deterministically between times of demand, such that the inventory evolves according to a Markov process in continuous time. The aim is to propose a way of finding optimal production schemes by minimizing a certain expected loss over some finite period. There are holding/production costs depending on the stock level, and random penalty amounts will occur due to excess demand which is assumed backlogged. For simplicity we will not incorporate fixed costs. We give some numerical illustrations.

*Keywords: Markov process, Martingale, Intensity process, Hilbert space, Quadratic loss, Markovian environment, Inflation model.*

## 1 Introduction

Minimiziation of costs in connection with inventory and production models is a major issue in logistics, and many model ideas and proposals have occurred over the years. Modeling the demand over periods as a random variable tends to more realistic assumptions but will also complicate the analysis significantly. A survey of the different ideas, discussions and solutions on inventory and production planning as it stands today, can e.g. be seen in [3], [7] and [8]. A frequently used model is the so called $(s, S)$ model, that is, when the stock level drops to $s$ or lower the inventory is raised to level $S$. Inventory models in the literature seem often centered around the $(s, S)$ terminology.

The aim of this paper is to contribute with solutions under a stochastic environment by using the advanced theory of point processes in continuous time. The theory is very powerful and can open for discussion of many highly complex problems. The basic idea is to model the demand for goods as a purely discrete stochastic process in continuous time. Between the times of demand production takes place (deterministically) with a continuous rate, such that the inventory (production minus demand) results in a Markov process in continuous time. For simplicity we assume that demand is governed by a compound Poisson process, but extensions to more complex structures are indeed possible. However, it has been statistically investigated at DDRE (Danish Defence Research Establishment), that the demand for spare parts for military vehicles can be modeled as a compound Poisson process, see e.g. [5] (in danish). Excess demand (negative inventory) is assumed to be backordered and we will not incorporate any aspects of lead times, such as e.g. time from production until the item is placed on the inventory. No backordering is when all excess demand is lost, and seems far more complex to describe and operate under in our setup. To not obscure the presentation we shall not incorporate any fixed (setup) costs.

In Section 2, we formulate the model and the main task is to find the formula for evaluating the expected cost over time, and this is based on the result of Theorem 2.1. The theorem is based on martingale theory which is a powerful tool for evaluating expected values for stochastic processes. In the rest of the section we analyse some examples and compare models of different complexity.

In Section 3, we briefly comment on an obvious extension on incorporating inflation in the cost process, but of course this paper is not complete and cannot cover all aspects on the subject. In Section 4 we give some numerical illustrations.

## 2 The model formulation

It is assumed throughout that all random variables are defined on some probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{R}$ and $\mathcal{R}_+$ denote the real line and the non-negative half line endowed with their usual Borel $\sigma$-algebras $\mathcal{B}$ and $\mathcal{B}_+$, respectively. For some stochastic process $X(\omega) = (X(\omega, t))_{t\geq 0}$, $\omega \in \Omega$, we say that $X$ belongs to the space $\mathcal{L}^2(\Omega \times [0, T])$, $T < \infty$, if

$$||X||_T^2 = E\left[\int_0^T X^2(s)ds\right] < \infty \,.$$

This space is a Hilbert space equipped with the inner product

$$< X, Y >_T = E\left[\int_0^T X(s)Y(s)ds\right],$$

and in particular it is equipped with the norm $||\ \ ||_T$. In the sequel we write $\mathcal{L}_T^2$ for $\mathcal{L}^2(\Omega \times [0, T])$.

We consider a situation where demand occurs at random times $T_1 < T_2 < \ldots$ The sequence $(T_n)_n$ is called a point process, and we let

$$N(t) = \sum_{n=1}^{\infty} \mathbf{1}(T_n \leq t),$$

denote the associated counting measure, where $\mathbf{1}(C)$ is the indicator of a set $C$. The amount of goods a customer demands at time $T_n$ is denoted $Z_n$ and is assumed to be a non-negative random variable. The sequence $(T_n, Z_n)_n$ is an example of a marked point process (MPP), where $Z_n$ is the mark associated to the point $T_n$. The process $X = (X(t))_{t\geq 0}$, with

$$X(t) = \sum_{n=1}^{N(t)} Z_n \,,$$

measures the total demand for goods over the period $[0, t]$. In the sequel we shall assume that $X$ is a compound Poisson process, that is, $N = (N(t))_{t\geq 0}$ is some Poisson process with continuous intensity $\lambda(t)$, and the sequence $(T_n)_n$ is independent of $(Z_n)_n$, where the $Z_n$'s are i.i.d., and we let $G(x)$, denote their common distribution function, and we use the convention $G(x) = 0$ for $x < 0$. Whenever the moments $E[Z_1]$ and $E[Z_1^2]$ exist, we recall the mean and variance (see e.g. [6])

$$E[X(t)] = E[Z_1]\int_0^t \lambda(s)ds \,, \quad Var[X(t)] = E[Z_1^2]\int_0^t \lambda(s)ds \,.$$

A general Markov process $I = (I(t))_{t\geq 0}$ for describing the inventory is then of the form

$$I(t) = I_0 + \int_0^t b(s, I(s))ds - X(t) \,, \tag{2.1}$$

where $I_0$ is the initial level, and $b : \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}_+$ is a continuous function playing the role of the production rate, and is allowed to depend on the current stock level, for instance if the inventory is 'high' one can lower the production and vice versa. Also if a constant rate of deterioration is relevant, it can be taken account for in the choice of $b(t, x)$, see below. Processes of the form (2.1) are extensively used in insurance and finance for modeling risk reserves and return rates on assets. It should be well-known that $I$ is a Markov process, that is, the $\sigma$-algebra $\sigma(I(s),\ s \geq t)$ is independent of the (history) $\sigma$-algebra $\sigma(I(s),\ s \leq t)$ for given $I(t)$, see e.g. [6].

Let $\tau_1 < \tau_2, \ldots$ be random times defined as $(\tau_0 = 0)$

$$\tau_n = \inf\{T_k \,|\, T_k > \tau_{n-1},\ I(T_k) < 0\} \,.$$

So $(\tau_n)_n$ is a thinning of $(T_n)_n$, and $\tau_n$ is the $n$th time $I$ jumps from either a positive to a negative value, or from a negative to a further negative value. These are obviously stopping times with respect to the natural filtration

$$\mathcal{F}_t^N = \sigma((T_n, Z_n)_n,\ T_n \le t),$$

that is, $\{\tau_n \le t\} \in \mathcal{F}_t^N$ for all $t \ge 0$, $n = 1, 2, \ldots$ The associated counting measure for $(T_n, Z_n)_n$ is given as

$$N(t, A) = \sum_{n=0}^{\infty} \mathbf{1}(T_n \le t,\ Z_n \in A)\,, \quad A \in \mathcal{B}_+\,,$$

and it is a standard result, that for any $\mathcal{F}_t^N$-predictable process $H(t, z)$ satisfying

$$E\left[\int_0^t \int_{\mathcal{R}_+} |H(s, z)|\lambda(s)G(dz)ds\right] < \infty\,, \tag{2.2}$$

for any $t \ge 0$, the process

$$M(t) = \int_0^t \int_{\mathcal{R}_+} H(s, z)(N(ds, dz) - \lambda(s)G(dz)ds)\,,$$

becomes an $\mathcal{F}_t^N$-martingale. A thorough presentation of martingales with respect to point process measures and other important aspects, can e.g. be found in [1] and [2]. Let

$$N^*(t) = \sum_{n=1}^{\infty} \mathbf{1}(\tau_n \le t)\,,$$

be the corresponding counting process for the point process $(\tau_n)_n$. For further analysis we must understand the $\mathcal{F}_t^N$-intensity process for $(\tau_n)_n$, which is (defined as) the process $\lambda^*(t)$, such that

$$M(t) = N^*(t) - \int_0^t \lambda^*(s)ds\,,$$

is an $\mathcal{F}_t^N$-martingale.

**Theorem 2.1** *The intensity process of $(\tau_n)_n$ exists, and is given by*

$$\lambda^*(t) = \lambda(t)\,[\overline{G}(I(t))\mathbf{1}(I(t) \ge 0) + \mathbf{1}(I(t) < 0)]\,, \quad t \ge 0,$$

*where $\overline{G}(x) = 1 - G(x)$.*

*Proof:* By definition, we have

$$\begin{aligned} N^*(t) &= \int_0^t \mathbf{1}(I(s) < 0)N(ds) \\ &= \int_0^t \int_{\mathcal{R}_+} \mathbf{1}(0 \le I(s-) < z)N(ds, dz) + \int_0^t \mathbf{1}(I(s-) < 0)N(ds)\,, \end{aligned}$$

and then by compensating, we get that

$$\begin{aligned} M(t) &= N^*(t) - \int_0^t \int_{\mathcal{R}_+} \mathbf{1}(0 \le I(s) < z)\,\lambda(s)G(dz)ds - \int_0^t \mathbf{1}(I(s) < 0)\,\lambda(s)ds \\ &= N^*(t) - \int_0^t \lambda(s)\,[\overline{G}(I(s))\mathbf{1}(I(s) \ge 0) + \mathbf{1}(I(s) < 0)]ds\,, \end{aligned}$$

becomes a martingale, since the mapping

$$t \to H(t, z) = \mathbf{1}(0 \le I(t-) < z) + \mathbf{1}(I(t-) < 0)$$

obviously is left-continuous (predictable) and satisfies (2.2). $\square$

When demand cannot be fulfilled we assume there will be a penalty cost, represented by non-negative random variables $Y_n$, $n = 1, 2, \ldots$, where $Y_n$ occurs at time $\tau_n$. The cost process is now introduced as

$$C(t) = \int_0^t c(s, I(s))ds + \sum_{n=1}^{N^*(t)} Y_n \,, \tag{2.3}$$

where $c : \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}_+$ is a continuous function such that $c(t, r)$ represents the rate of holding/production cost when the inventory is at level $r$ at time $t$. For simplicity we assume that $Y_1, Y_2, \ldots,$ are i.i.d. and independent of $(\tau_n)_n$, however this is not a crucial restriction. Assume that the mean $\mu = E[Y_1]$ exists. Then also the process

$$M(t) = \sum_{n=1}^{N^*(t)} Y_n - \mu \int_0^t \lambda^*(s)ds \,,$$

becomes an $\mathcal{F}_t^N$-martingale, and in particular for any $t \geq 0$, we have

$$E\left[\sum_{n=1}^{N^*(t)} Y_n\right] = E\left[\mu \int_0^t \lambda^*(s)ds\right].$$

Define the cost function

$$h(t, x) = c(t, x) + \mu\lambda(t)\left[\overline{G}(x)\mathbf{1}(x \geq 0) + \mathbf{1}(x < 0)\right], \quad x \in \mathcal{R}\,.$$

Let $T$ be some fixed finite period of time, and consider the expected cost $J(I) = E[C(T)]$, which from above reads

$$J(I) = E\left[\int_0^T h(s, I(s))ds\right],$$

and the task is now to control the $I$ process such that $J(I)$ is minimized. That means in particular that we must find the corresponding (optimal) choice of $b(t, x)$ (and $I_0$) in (2.1), since this function makes it possible to control the inventory between demands. We will not search among all functions, but consider subclasses for instance with $b(t, x)$ of the form $b(t, x) = \delta$ or perhaps $b(t, x) = \delta + \eta\, x$, for some constants $\eta$ and $\delta$, and the task would then be to find the optimal constants. Below we let $\mathcal{I}$ denote a set of processes $I$ corresponding to some functional class of $b(t, x)$. Evaluating $J(I)$ for some given $I$, is not an impossible task and can be done numerically via an integro-differential equation, confer e.g. Example 2.3 and Section 4 for ideas. However, as will appear below we will not try to minimize $J(I)$ directly.

Assume that we can find a $x_t^*$ satisfying

$$\inf_{x \in \mathcal{D}} h(t, x) = h(t, x_t^*)\,, \tag{2.4}$$

where $\mathcal{D}$ is some subset of $\mathcal{R}$. That is, $x_t^*$ minimizes for each $t$ the cost function $x \to h(t, x)$ subject to some constraints represented by $\mathcal{D}$ (for instance inventory cannot exceed a certain limit).

Obviously $\overline{G}(x)$ is a non-increasing function, and a natural assumption is to say that $x \to c(t, x)$ is increasing as $|x|$ increases, meaning that inventory costs increase with increasing stock level, and increasing backorders. This can give a behaviour of $h$ such that the existence of a $x_t^*$ is feasible, and perhaps a convex structure of $h$ occurs.

*Example 2.1.* As an illustration, assume that all $Z_i \equiv Q$, for some fixed integer $Q$, meaning that exactly $Q$ items are demanded each time. So $X(t) = Q\,N(t)$ for all $t$, and $G(x) = \mathbf{1}(x \geq Q)$. Assume furthermore that $c(t, x) = c\, x\mathbf{1}(x \geq 0)$, where $c > 0$. Hence

$$h(t, x) = c\, x\mathbf{1}(x \geq 0) + \mu\lambda(t)\left[\mathbf{1}(0 \leq x < Q) + \mathbf{1}(x < 0)\right],$$

and subject to $\mathcal{D} = [0, \infty)$, it is straightforward to check that $x_t^* = Q\,\mathbf{1}(c\,Q < \mu\lambda(t))$ satisfies (2.4). $\square$

To find an $I^* \in \mathcal{I}$ such that

$$J(I^*) = \inf_{I \in \mathcal{I}} J(I)$$

can quickly become rather complicated, and depends further on the structure of $h$. Instead we compromise by assuming that $x_t^* \in \mathcal{L}_T^2$, and introduce the loss

$$L(I) = E\left[\int_0^T (x_s^* - I(s))^2 ds\right], \quad I \in \mathcal{L}_T^2\,,$$

and shall find the optimal inventory process as the best in quadratic mean, that is

$$L(I^*) = \inf_{I \in \mathcal{I} \cap \mathcal{L}_T^2} L(I)\,.$$

Note that $I \in \mathcal{L}_T^2$ exactly when $E[Z_1^2] < \infty$. A simple example where an optimal solution exists and is easy to verify, is the following.

*Example 2.2.* Consider $I \in \mathcal{L}_T^2$ of the form

$$I(t) = I_0 + \delta\, t - X(t)\,,$$

and we are interested in finding the optimal production rate $\delta$ and initial inventory level $I_0$. Assume for simplicity that $\lambda(t)$ is independent of $t$, denoted $\lambda$, and define $\xi = E[Z_1]$. Then we recall that $E[X(t)] = \lambda\xi\, t$, and $Var[X(t)] = \lambda E[Z_1^2]\, t$. Since the function $x \to x^2$ is convex, the values of $\delta$ and $I_0$ minimizing

$$L(\delta, I_0) = E\left[\int_0^T (x_s^* - I_0 - \delta\, s + X(s))^2 ds\right],$$

are found from

$$\frac{\partial L}{\partial \delta} = 0\,, \qquad \frac{\partial L}{\partial I_0} = 0\,,$$

which give

$$\int_0^T s\,(x_s^* - I_0 - (\delta - \lambda\xi)s)\,ds = 0\,,$$

$$\int_0^T (x_s^* - I_0 - (\delta - \lambda\xi)s)\,ds = 0\,.$$

Therefore our optimal choices are the unique solutions satisfying

$$\begin{aligned}
\delta^* &= \lambda\xi + \frac{\int_0^T s\,(x_s^* - I_0^*)\,ds}{\int_0^T s^2\,ds} \\
&= \lambda\xi + \frac{3}{T^3}\int_0^T s\,x_s^*\,ds - \frac{3}{2T} I_0^*,
\end{aligned} \tag{2.5}$$

and

$$I_0^* = \frac{1}{T}\int_0^T x_s^*\,ds + (\lambda\xi - \delta^*)\frac{T}{2}\,. \tag{2.6}$$

Combining (2.5) and (2.6), we get

$$\begin{aligned}
I_0^* &= \frac{4}{T}\int_0^T x_s^*\,ds - \frac{6}{T^2}\int_0^T s\,x_s^*\,ds \\
&= \frac{6}{T^2}\int_0^T (\frac{2T}{3} - s)\,x_s^*\,ds,
\end{aligned}$$

and then

$$\delta^* = \lambda\xi + \frac{12}{T^3}\int_0^T (s - \frac{T}{2})\, x_s^*\, ds\,,$$

where we note that

$$\frac{6}{T^2}\int_0^T (\frac{2T}{3} - s)\, ds = 1.$$

and

$$\int_0^T (s - \frac{T}{2})\, ds = 0\,.$$

We see that $\delta^*$ and $I_0^*$ are going opposite each other, in the sense that, what causes $\delta^*$ to increase is causing $I_0^*$ to decrease and vice versa. The optimal inventory process is then

$$I^*(t) = I_0^* + \delta^*\, t - X(t)\,.$$

If $x_t^*$ becomes independent of $t$, denoted $x_0^*$ (it will whenever $h(t,x)$ is independent of $t$), we obtain

$$I_0^* = x_0^*, \quad \delta^* = \lambda\xi\,,$$

so the optimal production rate reduces to the average demand rate such that $E[I^*(t)] = x_0^*$, for all $t$. $\square$

A more realistic model could appear as

$$I(t) = I_0 + \int_0^t \delta(s) ds - X(t)\,,$$

with $\delta(t)$ piecewise constant over $[0,T]$,

$$\delta(t) = \sum_{k=1}^{K} \delta_k \mathbf{1}(t_{k-1} \le t < t_k)\,,$$

where $0 = t_0 < t_1 < \cdots < t_K = T$. The model opens for time varying effects in the production rate, perhaps there is no or a lower rate during night. Again the task is to find optimal values for $I_0$ and $\delta_i$, $i = 1, 2, \ldots, K$, by using the technique of Example 2.2. For $K \ge 2$ the calculations become rather formidable, and we shall omit the calculations. Another model is

$$I(t) = I_0 + \delta\, t - \eta\int_0^t I(s) ds - X(t)\,,$$

or in differential form

$$dI(t) = \delta\, dt - \eta\, I(t) dt - dX(t)\,,$$

where $\eta$ is some positive constant. So one reduces the production with a fixed proportion of $I(t)$ whenever $I(t)$ is non-negative, but speeds up the production with same proportion whenever there is excess demand. In the literature such a model is also used for describing a possible deterioration of goods, and $\eta$ is then interpreted as the rate of deterioration. We can solve this differential equation and get

$$I(t) = \exp(-\eta\, t)\left[I_0 + \delta\int_0^t \exp(\eta\, s) ds - \int_0^t \exp(\eta\, s) dX(s)\right],$$

which gives the mean

$$\begin{aligned} E[I(t)] &= \exp(-\eta\, t)\left[I_0 + \delta\int_0^t \exp(\eta\, s) ds - \xi\int_0^t \exp(\eta\, s)\lambda(s) ds\right] \\ &= \exp(-\eta\, t) I_0 + \frac{\delta}{\eta}(1 - \exp(-\eta\, t)) - \xi\int_0^t \exp(-\eta\,(t-s))\lambda(s) ds\,. \end{aligned}$$

So for this model we can also express the optimal solutions for $I_0$ and $\delta$ by repeating the techniques of Example 2.2. We shall omit the calculations.

Let us mention a rather complex case, where it does not seem possible to arrive at explicit expressions for the optimal parameters, but where numerical procedures must be used.

*Example 2.3.* We extend the model in Example 2.2 to the form

$$I(t) = I_0 + \delta\, t + \alpha \int_0^t \mathbf{1}(I(s) < \beta) ds - X(t)\,,$$

where we assume that $\alpha$ and $\beta$ are some fixed known constants, and our task is then to study the optimal choices of $\delta$ and $I_0$. For positive $\alpha$ the model tells that we increase the production with rate $\alpha$ whenever the inventory gets below $\beta$. We have

$$L(\delta, I_0) = E\left[\int_0^T (x_s^* - I_0 - \delta\, s - \alpha \int_0^s \mathbf{1}(I(u) < \beta) du + X(s))^2 ds\right],$$

and following the lines of Example 2.2, we find

$$\frac{\partial L}{\partial \delta} = 0\,, \quad \frac{\partial L}{\partial I_0} = 0\,,$$

are equivalent to

$$\int_0^T s\,(x_s^* - I_0 - \alpha \int_0^s P(I(u) < \beta) du - (\delta - \lambda\xi)s)\, ds = 0\,,$$

$$\int_0^T (x_s^* - I_0 - \alpha \int_0^s P(I(u) < \beta) du - (\delta - \lambda\xi)s)\, ds = 0\,,$$

or when using integration by parts

$$\int_0^T s\,(x_s^* - I_0 - (\delta - \lambda\xi)s)\, ds = \frac{\alpha}{2} \int_0^T (T^2 - s^2) P(I(s) < \beta) ds\,,$$

$$\int_0^T (x_s^* - I_0 - (\delta - \lambda\xi)s)\, ds = \alpha \int_0^T (T - s) P(I(s) < \beta) ds\,.$$

Writing $T^2 - s^2 = (T + s)(T - s)$, we can combine these equations to

$$\int_0^T (s - \frac{T}{2})\,(x_s^* - I_0 - (\delta - \lambda\xi)s)\, ds = \frac{\alpha}{2} \int_0^T s(T - s) P(I(s) < \beta) ds\,,$$

$$\int_0^T (x_s^* - I_0 - (\delta - \lambda\xi)s)\, ds = \alpha \int_0^T (T - s) P(I(s) < \beta) ds\,,$$

where we obviously can cancel $I_0$ in the first equation, giving

$$\int_0^T (s - \frac{T}{2})\,(x_s^* - (\delta - \lambda\xi)s)\, ds = \frac{\alpha}{2} \int_0^T s(T - s) P(I(s) < \beta) ds\,.$$

The probability $P(I(t) < \beta)$ will of course for any $t$ depend on $\delta$ and $I_0$, and can probably only be identified via an integro-differential equation. Namely define

$$F(t, s; q) = P(I(s) < \beta \,|\, I(t) = q)\,, \quad t \leq s, \quad q \in \mathcal{R}\,,$$

and let

$$q_y(t) = y + \delta\, t + \alpha \int_0^t \mathbf{1}(q_y(s) < \beta) ds\,, \quad y \in \mathcal{R},$$

be the deterministic integral curve that describes the paths of $I$ between its jumps. It can then be proved that $F$ satisfies the following backward integro-differential equation (see e.g. [4] for some ideas)

$$\frac{dF}{dt}(t,s;q_y(t)) = \lambda\, F(t,s;q_y(t)) - \lambda \int_{\mathcal{R}} F(t,s;q_y(t)-z)G(dz)\,, \quad t < s\,, \tag{2.7}$$

subject to initial condition $F(u,u;x) = \mathbf{1}(x<\beta)$, for all $u$ and $x$. Note that (2.7) is equivalent to the partial differential equation

$$\frac{\partial F}{\partial t}(t,s;q) + \frac{\partial F}{\partial q}(t,s;q)(\delta + \alpha\mathbf{1}(q<\beta)) = \lambda\, F(t,s;q) - \lambda \int_{\mathcal{R}} F(t,s;q-z)G(dz)\,, \quad t<s\,,$$

whenever the function $q \to F(t,s;q)$ is differentiable, but in general it is not.

Define now the conditional expectations

$$H_1(t;q) = \frac{\alpha}{2}\int_t^T s(T-s)F(t,s;q)ds\,,$$

$$H_2(t;q) = \alpha \int_t^T (T-s)F(t,s;q)ds\,.$$

Using (2.7) we can also arrive at the following (integro-differential) equations

$$\frac{dH_i}{dt}(t;q_y(t)) = \lambda\, H_i(t;q_y(t)) - \lambda \int_{\mathcal{R}} H_i(t;q_y(t)-z)G(dz) - L_i(t)\mathbf{1}(q_y(t)<\beta)\,, \quad t<T, \tag{2.8}$$

subject to initial condition $H_i(T;x) = 0$, for all $x$, and where

$$L_1(t) = \frac{\alpha}{2}\, t(T-t), \;\; L_2(t) = \alpha\,(T-t)\,.$$

In general it is not correct that $q_{y-z}(t) = q_y(t) - z$ (unless of course $\alpha = 0$), which could modify the differential equations to a system of ordinary differential equations indexed by $y$.

The optimal parameters can then be identified from

$$\int_0^T (s - \frac{T}{2})x^*_s\, ds + \frac{T^3}{12}(\lambda\xi - \delta) = H_1(0;I_0)\,, \tag{2.9}$$

$$\int_0^T x^*_s\, ds + \frac{T^2}{2}(\lambda\xi - \delta) = I_0\, T + H_2(0;I_0)\,. \tag{2.10}$$

So the procedure is to evaluate $H_1$ and $H_2$ over $[0,T]$ (starting at $T$), and then search for the values of $\delta$ and $I_0$, satisfying (2.9) and (2.10), confer Section 4. Recall that if $x^*_t$ is independent of $t$ then (2.9) reduces to

$$\frac{T^3}{12}(\lambda\xi - \delta) = H_1(0;I_0)\,.$$

□

Finally we shall introduce a Markovian environment such that we let the production rate depend on the state of a Markov process $\Theta = (\Theta(t))_{t\geq 0}$, with finite state space denoted $\mathcal{J}$. This makes it possible to take account for different stochastic phenomena, such as the life time of a machine or a breakdown state where production is not available. As a simple illustration we assume that $I$ is modified to the form

$$I(t) = I_0 + \int_0^t \delta_{\Theta(s)} ds - X(t)\,,$$

where $X(t)$ is as above, and $\delta_i$, $i = 1,2,\ldots,\mathcal{J}$ are the rates of production. We shall assume that $X$ is independent of $\Theta$. Let

$$Q_{ij}(s,t) = P(\Theta(t) = j\,|\,\Theta(s) = i)\,, \quad i,j \in \mathcal{J}\,, \quad s \leq t\,,$$

denote the transition probabilities, and we shall assume that $\Theta$ is given by transition intensities

$$\lambda_{ij}(t) = \lim_{h \searrow 0} \frac{Q_{ij}(t, t+h)}{h}, \quad i \neq j.$$

Define $R_j(t) = \int_0^t \mathbf{1}(\Theta(s) = j) ds$, the exposure time in state $j$ over $[0, t]$. We write $I(t)$ as

$$I(t) = I_0 + \sum_{i \in \mathcal{J}} \delta_i R_i(t) - X(t),$$

and the task is to minimize

$$L(I_0, (\delta_i)_{i \in \mathcal{J}}) = E\left[\int_0^T (x_s^* - I_0 - \sum_{i \in \mathcal{J}} \delta_i R_i(s) + X(s))^2 ds\right],$$

to find the optimal choices of $I_0$ and the $\delta_i$. The complexity is of a level to the case with piecewise constant intensities above, except that we here have stochastic exposure times $R_i(t)$. Let us consider a simple case.

*Example 2.4.* Assume that $\mathcal{J} = \{0, 1\}$ consists of only two states, and the process starts out in state 0, and put $\delta_1 = 0$. State 1 could represent some breakdown state where production ceases temporarily or permanently if $\lambda_{10}(t) = 0$ for all $t$, and in the latter, $\Theta$ then describes a life period for the production. We have $E[R_0(t)] = \int_0^t Q_{00}(0, s) ds$. With $\lambda$ and $\xi$ as above, the equations to evaluate the optimal parameters become

$$\int_0^T E[R_0(s)\,(x_s^* - I_0 - \delta_0\, R_0(s) + \lambda\xi\, s)]\, ds = 0,$$

$$\int_0^T (x_s^* - I_0 - \delta_0 E[R_0(s)] + \lambda\xi\, s)\, ds = 0,$$

which give

$$\delta_0^* = \frac{\lambda\xi \int_0^T s E[R_0(s)]\, ds}{\int_0^T E[R_0^2(s)]\, ds} + \frac{\int_0^T E[R_0(s)]\,(x_s^* - I_0^*)\, ds}{\int_0^T E[R_0^2(s)]\, ds},$$

and

$$I_0^* = \frac{1}{T}\int_0^T x_s^*\, ds + \lambda\xi \frac{T}{2} - \frac{\delta_0^*}{T}\int_0^T E[R_0(s)]\, ds.$$

Assuming that $x_t^*$ is independent of $t$, denoted by $x_0^*$, we can arrive at the following expressions

$$\delta_0^* = \frac{\lambda\xi \int_0^T (s - \frac{T}{2}) E[R_0(s)]\, ds}{\int_0^T E[R_0^2(s)]\, ds - \frac{1}{T}(\int_0^T E[R_0(s)]\, ds)^2}$$

and

$$I_0^* = x_0^* + \lambda\xi\left[\frac{T}{2} - \frac{\frac{1}{T}\int_0^T E[R_0(s)]\, ds \int_0^T (s - \frac{T}{2}) E[R_0(s)]\, ds}{\int_0^T E[R_0^2(s)]\, ds - \frac{1}{T}(\int_0^T E[R_0(s)]\, ds)^2}\right].$$

□

# 3 A model with inflation

There were no dynamic behaviour in the common distribution function for the penalty cost, so we cannot take account for e.g. inflation. However, if we assume there is a constant annual interest, and let $\eta$ denote the annual force of interest, we can replace the jump part in (2.3) with the discounted process $\sum_{n=1}^{N^*(t)} \exp(-\eta\,\tau_n)Y_n$, and the mean now becomes

$$E\left[\sum_{n=1}^{N^*(t)} \exp(-\eta\,\tau_n)Y_n\right] = E\left[\mu \int_0^t \exp(-\eta\,s)\lambda^*(s)ds\right].$$

So in a model with constant inflation of rate $\eta$, we replace $h$ with

$$h(t,x) = c(t,x) + \mu\lambda(t)\exp(-\eta\,t)\left[\overline{G}(x)\mathbf{1}(x \geq 0) + \mathbf{1}(x < 0)\right].$$

# 4 Some numerical illustrations

The aim of this section is to indicate how (2.7) and (2.8) can be solved numerically, and hence how models of such complexity are useful in practical life.

Assume that $G$ is degenerated in a point $z_0$, that is $G(x) = \mathbf{1}(x \geq z_0)$, and that $x_t^* = z_0$, which is a possible solution as mentioned in Example 2.1. Then (2.8) reduces to

$$\frac{dH_i}{dt}(t;q_y(t)) = \lambda\,H_i(t;q_y(t)) - \lambda\,H_i(t;q_y(t)-z_0) - L_i(t)\mathbf{1}(q_y(t) < \beta)\,,$$

which we will solve numerically using standard methods (Runge-Kutta) for ordinary differential equations, and find the corresponding values of $I_0^*$ and $\delta^*$ satisfying

$$\frac{T^3}{12}(\lambda z_0 - \delta^*) = H_1(0;I_0^*)\,,$$

$$z_0\,T + \frac{T^2}{2}(\lambda z_0 - \delta^*) = I_0^*\,T + H_2(0;I_0^*)\,,$$

as found from (2.9) and (2.10). It is tempting to say that $I_0^*$ is close to $z_0$, and of course if $\beta = \infty$, we are then back in the simple model in Example 2.2. So intuitively, the larger $\beta$ is chosen the closer $I_0^*$ becomes to $z_0$, and the closer $\delta^*$ becomes to $\lambda z_0 - \alpha$. On the other hand, $\beta$ should not be chosen too 'small', which would imply that $F$ and hence $H_i$ become small, and then $\alpha$ has no significant effect. Put

$$K_1(y) = H_1(0;y) - \frac{T^3}{12}(\lambda z_0 - \delta)\,,$$

and

$$K_2(y) = H_2(0;y) + (y - z_0)T - \frac{T^2}{2}(\lambda z_0 - \delta)\,.$$

Note that $y \to H_i(0;y)$ are decreasing functions, and it will appear that $K_2(y)$ is increasing.

Table 3.1. $z_0 = 1 \quad \delta = 7.115 \quad \alpha = 3 \quad \lambda = 10 \quad \beta = 5 \quad T = 1$

| $y$ | $K_1(y)$ | $K_2(y)$ |
|---|---|---|
| 0.0 | 0.00659 | −0.95061 |
| 0.1 | 0.00618 | −0.85178 |
| 0.2 | 0.00571 | −0.75311 |
| 0.3 | 0.00519 | −0.65460 |
| 0.4 | 0.00462 | −0.55628 |
| 0.5 | 0.00397 | −0.45817 |
| 0.6 | 0.00326 | −0.36029 |
| 0.7 | 0.00248 | −0.26266 |
| 0.8 | 0.00161 | −0.16531 |
| 0.9 | 0.00065 | −0.06825 |
| 0.925 | 0.00040 | −0.04404 |
| 0.95 | 0.00014 | −0.01985 |
| 0.975 | −0.00013 | 0.00431 |
| 1.0 | −0.00041 | 0.02845 |
| 1.1 | −0.00156 | 0.12480 |

We see that $K_i(y)$, $i = 1, 2$, switches sign somewhere between 0.95 and 0.975, so $I_0^*$ lies between these values and is close to 0.975, and we find our corresponding choice for $\delta^*$ equal to 7.115. Note that $I_0^*$ is close to $z_0$, and that $\delta^*$ is close to $\lambda z_0 - \alpha$ (confer comments above).

Table 3.2. $z_0 = 1 \quad \delta = 8.832 \quad \alpha = 3 \quad \lambda = 10 \quad \beta = 2 \quad T = 1$

| $y$ | $K_1(y)$ | $K_2(y)$ |
|---|---|---|
| 0.0 | 0.04686 | −0.48910 |
| 0.1 | 0.04106 | −0.41791 |
| 0.2 | 0.03509 | −0.34844 |
| 0.3 | 0.02896 | −0.28061 |
| 0.4 | 0.02268 | −0.21467 |
| 0.5 | 0.01627 | −0.15050 |
| 0.6 | 0.00972 | −0.08838 |
| 0.7 | 0.00306 | −0.02816 |
| 0.725 | 0.00138 | −0.01352 |
| 0.75 | −0.00031 | 0.00104 |
| 0.8 | −0.00370 | 0.02983 |
| 0.9 | −0.01053 | 0.08579 |

From Table 3.2 we find $I_0^*$ close to 0.75, which is smaller than for the previous case with $\beta = 5$, and $\delta^*$ is now found equal to 8.832.

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# References


[1] P.K. Andersen et al. (1993), *Statistical Models Based on Counting Processes*, Springer-Verlag.

[2] P. Brémaud (1981), *Point Processes and Queues*, Springer-Verlag.

[3] S.C. Graves, A.H.G. Rinnooy Kan and P.H. Zipkin (1993), *Logistics of Production and Inventory*, North-Holland.

[4] C.M. Møller (1995), Stochastic Differential Equations for Ruin Probabilities, *J. Appl. Prob.* **32**, 74-89.

[5] C.M. Møller and S. Livbjerg (1999), Statistisk Analyse af Reservedelsforbruget for Hærens Køretøjer (Danske forhold), *Research Report M-23/1999, Danish Defence Research Establishment.*

[6] D.F. Snyder and M.I. Miller (1991), *Random Point Processes in Time and Space*, Springer-Verlag.

[7] E.A. Silver, D.F. Pyke and R. Peterson (1998), *Inventory Management and Production Planning and Scheduling*, John Wiley & Sons.

[8] H.C. Tijms (1994), *Stochastic Models: An Algorithmic Approach*, John Wiley & Sons.